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SAMPLING INSPECTION BY VARIABLES: NONPARAMETRIC SETTING

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Abstract A classic statistical problem is the optimal construction of sampling plans to accept or reject a lot based on a small sample. We propose a new asymptotically optimal solution for the acceptance sampling by variables setting where we allow for an arbitrary unknown underlying distribution. In the course of this, we assume that additional sampling information is available, which is often the case in real applications. That information is given by additional measurements which may be affected by a calibration error. Our results show that, firstly, the proposed decision rule is asymptotically valid under fairly general assumptions. Secondly, the estimated optimal sample size is asymptotically normal. Further, we illustrate our method by a real data analysis and we investigate to some extent its finite sample properties and the sharpness of our assumptions by simulations.

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1. INTRODUCTION

Acceptance sampling is concerned with the (optimal) construction of sampling plans to accept or reject a possibly large lot based on a small sample. This can be based on quality characteristics with two outcomes (sampling by attributes), or characteristics measured on a metric scale (sampling by variables). The classical procedures for the latter problem assume normally distributed observations, which is too restrictive for many applications. Thus, in this article we consider the problem to construct asymptotically optimal sampling plans for acceptance sampling by variables for an arbitrary unknown underlying distribution, when additional sampling information is available. That additional information is given by additional measurements of the lot or a further random sample from the production process, where these measurements may be affected by calibration errors. This situation arises often in real applications. E.g., suppliers to industry usually deliver in lots, and these lots are checked using acceptance sampling rules. As long as the distribution does not change, one may pool the data from these analyses. Our work is also motivated by a specific quality control problem dealing with photovoltaic modules. Here, due to technical reasons, the distribution of these measurements varies considerably from lot to lot, and is usually non-normal. However, producers sometimes hand out their lot measurements. Although these measurements are often affected by a (possibly advisedly arranged) calibration error, they can be used to estimate the shape of the underlying distribution. For details we refer to [7].

Denote by $X_1, \dots X_N$ a quality characteristic from a random sample from the production process. The common distribution function of the X_i's will be denoted by F. We assume throughout that the fourth moment of F exists, and let μ and σ^2 denote the mean and variance of F, respectively. Further assumptions will be given in the next section. Item i is called conforming (non-defect), if $X_i > \tau$ for some $\tau \in \mathbb{R}$. Then the fraction of non-conforming (defect) items is given by the probability

$$p := \mathbb{E}_{(\mu,\sigma^2)}\left[\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{\{X_i\le\tau\}}\right] = \mathbb{P}_{(\mu,\sigma^2)}[X_1 \le \tau], \tag{1.1}$$

and is the common quantity to define the quality of a lot. Here and in the sequel $\mathbf{1}_A$ denotes the indicator function of the event A. The lot should be accepted if p is smaller than the acceptable quality level (AQL) p_α and rejected if p is larger than the rejectable quality level (RQL) p_β. The subscripts α and β will be explained below. p_α and p_β have to be settled by an agreement between the producer and the consumer of the shipment. In any case we wisely assume $p_\alpha < p_\beta$.

The classic approach to the problem is to construct a decision rule such that the error probabilities of a false acceptance and a false rejection of the lot are controlled. For an overview see the monograph [10], further information can be found in, e.g., [2, 3] and in references cited therein. For normally distributed observations the problem is straightforward and well known. For instance, sampling plans for known variance, unknown variance, and average range, which are based on the Rao-Blackwell estimator of the fraction of non-conforming items, have been developed in [8]. Further plans for non-normal distributions have been studied, but only for certain parametric models where the form of the underlying distribution is known and the proportion of non-conforming items is a simple function of the parameters. For instance, [18] studied inspection by variables based on Burr's distribution. When the distribution is known up to a location parameter, the problem has been studied by [4, 16]. In this paper, we propose a nonparametric solution for the general case. It is based on nonparametric estimators of all unknown quantities appearing in the formulae that specify the sampling plan (in [4, 16]) when the normality assumption is dropped. We provide a rigorous asymptotic justification for the proposed decision rule, which relies on the delta method for Hadamard differentiable statistical functionals and on a functional central limit theorem for empirical processes with estimated parameters. Background on the latter tools is given in [1, 12, 17], for instance.

The organization of the rest of the paper is as follows. In Section 2 the distributional model and the proposed decision rule are carefully introduced, and the assumptions required by our asymptotic result are discussed. Section 3 provides the main result. We prove an asymptotic representation of the operating characteristic, which justifies a simple plug-in rule for the choice of the sample size. The asymptotic behavior of that estimated sample size is also investigated in terms of a central limit theorem. Section 4 provides a real data analysis and results from an extensive simulation study. We analyze real photovoltaic measurements and illustrate our solution to the problem. The data analysis shows that there is indeed need for nonparametric acceptance sampling procedures. Our simulations provide valuable insight into the applicability of the method. The results indicate that, firstly, the proposed method works reliable in many situations of practical interest. Secondly, it provides evidence that the fourth moment assumptions required by our theoretical results can not be weakened.

The proofs of the main results are given in Section 5. Several appendices provide further technical details and a brief review of the functional delta method used in our proofs.

2. ASSUMPTIONS, MODEL, AND THE DECISION RULE

Clearly, the true fraction of non-conforming items p, which equals the expectation of the fraction of non-conforming items of the lot of size N, is unknown. Since an inspection of all lot items is usually not feasible, the consumer's decision to accept or reject the shipment has to be based on $n < N$ i.i.d. control measurements, $X_1', \dots, X_n'$, which are also distributed according to F. We aim at defining a suitable decision rule based on $X_1', \dots, X_n'$ to accept or reject the lot. The additional information is given by a further i.i.d. random sample $X_1^\Delta, \dots, X_m^\Delta$ of size m representing, for instance, a data sheet from the producer. The values X_i^Δ may be affected by a calibration error Δ relative to the control measurements. That is, we just require $X_i^\Delta \stackrel{d}{=} X_j + \Delta$, for some $\Delta \in \mathbb{R}$ and all i, j.

We emphasize that the samples (X_i') and (X_i^Δ) are not required to be independent, and we assume $1 \ll n \ll m \le N$. We will refer to the basic probability space as $(\Omega, \mathcal{F}, \mathbb{P}_{(\mu,\sigma^2)})$ and assume that it is rich enough to host whole sequences (X_i') and (X_i^Δ) as i.i.d. random variables with distribution function F.

Concerning F, we do not assume any parametric form, but only require that F is a member of the nonparametric location-scale family

$$\mathbb{F} := \left\{ F_{(\mu,\sigma^2)}(.) := G((. - \mu)/\sigma) \ : \ (\mu, \sigma^2) \in \mathbb{R} \times (0, \infty) \right\},$$

where G $(= F_{(0,1)})$ is an arbitrary but unknown distribution function.

Before proceeding with the exposition of the statistical problem and the proposed solution, let us phrase and discuss our rather weak assumptions on the distribution function G.

Assumption 2.1 *$\int_{\mathbb{R}} x dG(x) = 0$, $\int_{\mathbb{R}} x^2 dG(x) = 1$, and $\int_{\mathbb{R}} x^4 dG(x) < \infty$.*

Assumption 2.2 *G is continuously differentiable, and*

$$\left\| F_{(\mu,\sigma^2)+(v_1,v_2)}(.) - F_{(\mu,\sigma^2)}(.) - [\dot{F}_{(\mu,\sigma^2)}(.)](v_1, v_2)' \right\|_\infty = o(|v|) \tag{2.1}$$

for every fixed $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$, where $(v_1, v_2)'$ is the transpose of $v = (v_1, v_2)$, and

$$\dot{F}_{(\mu,\sigma^2)}(t) := \left(\frac{\partial}{\partial \mu} F_{(\mu,\sigma^2)}(t), \frac{\partial}{\partial \sigma^2} F_{(\mu,\sigma^2)}(t) \right) = -\frac{1}{\sigma} \left(G'\Big(\frac{t-\mu}{\sigma}\Big), \frac{t-\mu}{2\sigma^2}\, G'\Big(\frac{t-\mu}{\sigma}\Big) \right). \tag{2.2}$$

Assumption 2.3 *G is strictly increasing on $[a, b]$, where $a := \sup\{t \in \mathbb{R} : G(t) = 0\}$ and $b := \inf\{t \in \mathbb{R} : G(t) = 1\}$ with the conventions $\sup \emptyset := -\infty$ and $\inf \emptyset := \infty$.*

Here, $\|.\|_\infty$ denotes the usual supremum norm. Assumption 2.3 implies in particular that the map $(\mu, \sigma^2) \mapsto F_{(\mu,\sigma^2)}$ is injective. Assumption (2.1) just means that the mapping $(\mu, \sigma^2) \mapsto F_{(\mu,\sigma^2)}$ (from $\mathbb{R} \times (0, \infty)$ to the space of all cádlàg functions on $\bar{\mathbb{R}}$ equipped with $\|.\|_\infty$) is Fréchet differentiable at (μ, σ^2) with Fréchet derivative $\dot{F}_{(\mu,\sigma^2)}$. The notion of Fréchet differentiability and the definition of cádlàg functions can be found, for instance, in [17, p.297 and p.257]. Assumption 2.2 is quite an abstract condition, and therefore we mention a more transparent sufficient condition. If G is twice continuously differentiable and satisfies

$$\int_{\mathbb{R}} |x|\, |G''(x)| dx < \infty, \tag{2.3}$$

then (2.1) holds, cf. Lemma A.1 in the Appendix A. With the help of this sufficient condition one can easily check that for instance the normal distribution satisfies Assumption 2.2 (apart from Assumptions 2.1 and 2.3). Further examples for G satisfying (2.3) are all twice continuously differentiable distribution functions with compact support or Pareto-type tails $\kappa|x|^{-\gamma}$ (for x away from 0) with $\gamma > 1$ and suitable $\kappa = \kappa_\gamma > 0$.

To motivate our proposal, let us first consider the treatment of the acceptance by variables problem when σ and G are known (this condition will be skipped below). From an intuitive point of view, the lot should be accepted if and only if

$$\sqrt{n}\,\frac{\bar{X}'_n - \tau}{\sigma} > c, \tag{2.4}$$

where c is some suitable positive constant and $\bar{X}'_n$ is the sample mean of $X'_1, \dots, X'_n$. The free parameters n and c determine the sample size and the acceptance range, respectively. The consumer is obviously interested in a small sample size n. This interest is however foiled by the problem that a small n does not admit a safe decision. Intuitively, a small sample size n cannot ensure that a decision rule keeps the probability of the type II error (consumer risk) small. On the other hand, the producer will typically insist on a small probability of the type I error (producer risk). Recall that the type I error paraphrases a rejection of the lot although it is of high quality. In contrast, the type II error paraphrases an acception of the shipment although it is of low quality. A fair decision rule should keep both the type I error and the type II error small. That is, for $0 < \beta < \alpha$,

$$p \le p_\alpha \quad \Rightarrow \quad \text{prob}\,[\text{“acceptance”}] \ge \alpha, \tag{2.5}$$

$$p \ge p_\beta \quad \Rightarrow \quad \text{prob}\,[\text{“acceptance”}] \le \beta. \tag{2.6}$$

The values α and β determine the error probabilities the contracting parties are willing to accept. Hence, α and β fix the confidence level of the decision rule. In particular, they should be part of the contract to supply the shipment.

The basic question is how small n may be in order to gurantee (2.5)-(2.6). Toward an answer we note that the test statistic on the left-hand side of (2.4) can be written as the sum of $\sqrt{n}(\mu - \tau)/\sigma$ and an expression that is asymptotically standard normal. By (1.1) we also have $p = G((\tau-\mu)/\sigma)$, where $G(t) = F(\sigma t+\mu)$ denotes the distribution function of the standardized variable $(X_1 - \mu)/\sigma$. Thus, if G is strictly increasing and n is reasonably large, we have

$$\text{prob}\left[\text{“acceptance”}\right] \approx 1 - \Phi\left(c + \sqrt{n}\, G^{-1}(p)\right) \tag{2.7}$$

with Φ the distribution function of the standard normal distribution and $G^{-1}(p) = \inf\{t \in \mathbb{R} : G(t) \geq p\}$ the p-quantile of G. From (2.7) one easily deduces (cf. Appendix B) the minimal sample size n along with the acceptance threshold c such that (2.5)-(2.6) is ensured:

$$n = \left(\frac{\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta)}{G^{-1}(p_\beta) - G^{-1}(p_\alpha)} \right)^2 \tag{2.8}$$

$$c = \Phi^{-1}(1-\alpha) - \sqrt{n}\, G^{-1}(p_\alpha). \tag{2.9}$$

Strictly speaking, (2.5)-(2.6) are ensured only approximately. Note that c in (2.9) remains unchanged when α is replaced by β. This is a by-product of the derivation of (2.8)-(2.9) in the Appendix B.

A natural way to handle the unknown quantities σ and G is to plug in appropriate estimators. Here the additional sample $X_1^\Delta, \dots, X_m^\Delta$ comes into play. The superscript indicates that it may be shifted by a constant relative to the control measurements, but the following estimators of σ and G are location invariant, i.e., robust against shifts.

$$\sigma_m := \left(\frac{1}{m-1} \sum_{i=1}^m \left(X_i^\Delta - \bar{X}_m^\Delta \right)^2 \right)^{1/2}, \tag{2.10}$$

$$G_m(t) := \frac{1}{m} \sum_{i=1}^m \mathbf{1}_{(-\infty,t]}\left((X_i^\Delta - \bar{X}_m^\Delta)/\sigma_m \right) \qquad (t \in \mathbb{R}). \tag{2.11}$$

Note that G_m is the empirical distribution function of the (empirically) standardized random variables $(X_1^\Delta - \bar{X}_m^\Delta)/\sigma_m, \dots, (X_m^\Delta - \bar{X}_m^\Delta)/\sigma_m$. We now replace the left-hand side of (2.4) by

$$T_n := \sqrt{n}\, \frac{\bar{X}_n' - \tau}{\sigma_m}$$

to obtain the following decision rule:

RULE 2.1 *The lot will be accepted if and only if $T_n > c$.*

Now intuition suggests to use the formulae (2.8) and (2.9) with G replaced by G_m. We postpone a more detailed derivation and a rigorous justification of this idea to the next section.

For that main result we need the following assumption that the sample size m is essentially larger than n, i.e.,

$$m = m_n \qquad \text{and} \qquad \lim_{n\to\infty} n/m = 0. \tag{2.12}$$

This assumption is indeed crucial. If m and n grew at the same rate, i.e. if $\lim_{n\to\infty} n/m$ existed in $(0,\infty)$, one might still obtain versions of (2.8) and (2.9), but belike they would depend on G which was pointless. For this obvious reason we impose (2.12), and, as indicated in the introduction, that assumption is not restrictive for applications.

3. MAIN RESULT

Recall that our objective is to determine the acceptance range, i.e., the threshold c, and the minimum sample size n of Rule 2.1 such that (2.5)-(2.6) is ensured. The specification of (n,c) is given in (3.6)-(3.7) which, to some extent, is the main result of this article. Toward the justification we first rewrite (2.5)-(2.6). To this end we introduce the operation characteristic (or acceptance probability function)

$$\mathcal{A}_{n,c}(p) := \mathbb{P}_{(\mu,\sigma^2)}[T_n > c] \qquad (p \in [0,1]). \tag{3.1}$$

Recalling $p = G((\tau-\mu)/\sigma)$ and taking Assumption 2.3 into account we deduce a one-to-one correspondence between p and $\mu = \mu(p)$, so that the right-hand side of (3.1) can indeed be seen as a function of p. Now we can rewrite (2.5)-(2.6) as

$$p \le p_\alpha \quad \Rightarrow \quad \mathcal{A}_{n,c}(p) \ge \alpha, \tag{3.2}$$

$$p \ge p_\beta \quad \Rightarrow \quad \mathcal{A}_{n,c}(p) \le \beta. \tag{3.3}$$

As we did not restrict F to be any specific parametric distribution, we have no option but require (3.2)-(3.3) only asymptotically ($n\to\infty$). The key for the formulae (3.6) and (3.7) below is the following main theorem.

THEOREM 3.1 *Suppose (2.12) and Assumptions 2.1-2.3 hold. Then, for every $p \in (0,1)$, there exists a sequence $(\delta_n(p))$ of random variables converging in $\mathbb{P}_{(\mu,\sigma^2)}$-probability to 0 and satisfying for all $c \in \mathbb{R}$,*

$$\mathcal{A}_{n,c}(p) = \mathbb{P}_{(\mu,\sigma^2)}\left[\sqrt{n}\,\frac{\bar{X}'_n-\mu}{\sigma} + \delta_n(p) \;\ge\; c + \sqrt{n}\; G_m^{-1}(p)\right]. \tag{3.4}$$

The proof is postponed to Section 5. According to Theorem 3.1 we have for reasonably large n the analogue of (2.7),

$$\mathcal{A}_{n,c}(p) \approx 1 - \Phi\left(c + \sqrt{n}\, G_m^{-1}(p)\right). \tag{3.5}$$

Indeed, the left-hand side of the event in (3.4) is asymptotically standard normal by the central limit theorem and Slutsky's lemma. Now we may proceed as in the Appendix B, with G replaced by G_m, to obtain the (asymptotically) optimal sampling plan, i.e., (n, c) with the minimal n satisfying (3.2)-(3.3):

$$n = n_m = \left(\frac{\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta)}{G_m^{-1}(p_\beta) - G_m^{-1}(p_\alpha)}\right)^2 \tag{3.6}$$

$$c = c_m = \Phi^{-1}(1-\alpha) - \sqrt{n}\, G_m^{-1}(p_\alpha). \tag{3.7}$$

Note that c in (3.7) remains unchanged when α is replaced by β. This is a by-product of the considerations in the Appendix B. Our analysis suggests that the sample size n should be chosen as the smallest integer being larger than the right-hand side of (3.6).

At first glance, formula (3.6) seems to be pointless. It provides a formula for n but the right-hand side implicitly depends on n since we presupposed (2.12). Recall, however, that there are many practical situations where $X_1^\Delta, \ldots, X_m^\Delta$, and therefore G_m and σ_m, are a priori known for a given "sufficiently large" m, but the shift Δ is unknown. This is for instance exactly the case with our motivating example where the producer hands out a data sheet. We also emphasize that our asymptotic framework is by all means justified, at least for a stringent choice of α and β. More precisely, if p_α and p_β are fixed and if $\alpha \nearrow 1$ or $\beta \searrow 0$ then $(\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta))$ approaches $-\infty$, so that n in (3.6) tends to $+\infty$ since $(G_m^{-1}(p_\beta) - G_m^{-1}(p_\alpha))$ remains bounded with high probability. Note that the latter expression converges in probability to the constant $(G^{-1}(p_\beta) - G^{-1}(p_\alpha))$, as $m \to \infty$, by Lemma 5.2.

The following theorem provides an intuition of the asymptotic behavior ($m \to \infty$) of the sample size $n = n_m$ specified by (3.6). The proof is postponed to Section 5.

THEOREM 3.2 *Let n_m and n_∞ be defined by the right-hand sides of (3.6) and (2.8), respectively. Then, under $\mathbb{P}_{(\mu,\sigma^2)}$,*

$$\sqrt{m}\,(n_m - n_\infty) \xrightarrow{d} N(0, V(\alpha, \beta, p_\alpha, p_\beta)), \tag{3.8}$$

as $m \to \infty$*, where*

$$V(\alpha, \beta, p_\alpha, p_\beta) := 4\,\frac{(\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta))^4}{(G^{-1}(p_\beta) - G^{-1}(p_\alpha))^6}\;\tilde{V}(p_\alpha, p_\beta) \tag{3.9}$$

with

$$\tilde{V}(p_\alpha, p_\beta) := \frac{\Gamma(G^{-1}(p_\beta), G^{-1}(p_\beta))}{G'(G^{-1}(p_\beta))^2} - \frac{2\,\Gamma(G^{-1}(p_\alpha), G^{-1}(p_\beta))}{G'(G^{-1}(p_\alpha))G'(G^{-1}(p_\beta))} + \frac{\Gamma(G^{-1}(p_\alpha), G^{-1}(p_\alpha))}{G'(G^{-1}(p_\alpha))^2}.$$

Here Γ *is a symmetric function on* $\mathbb{R} \times \mathbb{R}$ *defined by*

$$\Gamma(s,t) := G(s)(1 - G(t)) - \dot{G}(t)b(s) - \dot{G}(s)b(t) + \dot{G}(s)A\dot{G}(t)'$$

for $s \le t$*, where*

$$\dot{G}(t) := -\left(G'(t),\ (t/2)\,G'(t)\right)$$

$$b(t) := \begin{pmatrix} \mathbb{E}_{(0,1)}\left[X_1 \mathbf{1}_{(-\infty,t]}(X_1)\right] \\ \mathbb{E}_{(0,1)}\left[(X_1^2 - 1)\mathbf{1}_{(-\infty,t]}(X_1)\right] \end{pmatrix}$$

$$A := \begin{pmatrix} 1 & \mathbb{E}_{(0,1)}[X_1^3] \\ \mathbb{E}_{(0,1)}[X_1^3] & \mathbb{E}_{(0,1)}[X_1^4] - 1 \end{pmatrix}.$$

The theorem shows that $\sqrt{m}\,(n_m - n_\infty)$ is asymptotically normal. The good thing is that we have specified the asymptotic variance explicitly. Unfortunately the formula is rather complicated. We can see at least that the asymptotic variance is independent of (μ, σ^2). Moreover, the power six of the denominator on the right-hand side of (3.9) indicates that the fluctuation of n_m might be relatively strong even for large values of m. This phenomenon can indeed be observed in the table given in Subsection 4.2. At this point we emphasize that $(G^{-1}(p_\beta) - G^{-1}(p_\alpha))$ is typically smaller than 1, and that $|\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta)|$ is typically larger than 1. If, for example, $1 - \alpha = \beta = 0.05$, AQL $p_\alpha = 0.01$, RQL $p_\beta = 0.05$ and $G = \Phi$, then $(G^{-1}(p_\beta) - G^{-1}(p_\alpha)) \approx 0.6815$ and $|\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta)| \approx 3.2897$.

4. Example and Simulations

In this section, we illustrate the proposed method by a real data example and study the statistical properties by a Monte Carlo study. The latter addresses two issues. First, we were interested in the finite sample behavior of the procedure when applied to realistic sample sizes and distributional models of practical relevance. Second, we analyzed to some extent whether the assumptions of our limit theorem are sharp. Here the fourth moment assumption is of primary concern, because it rules out distributions with thick tails. Recall that the thickness of the tails is often measured by the (empirical) kurtosis, i.e., the (empirical) fourth moment after (empirical) standardization.

4.1. *Example*

Our work is motivated by a project with TÜV Rheinland Immissionsschutz und Energiesysteme GmbH, which offers quality control services for producers of photovoltaic modules. We applied the procedure to real power measurements under so-called standard conditions. The producer handed out a so-called flasher list of $m := N = 500$ measurements with $\sigma_m = 4.23$, a subsample of a larger list. The Shapiro test for normality yielded a p-value of < 0.0001. Figure 1 depicts a nonparametric density estimate of the these measurements. We used a kernel density estimator with Gaussian kernel and bandwidth choice by cross-validation. The distribution is apparently non-normal and asymmetric, having several modes. The bump around 175 is not an artifact and also present in the larger flasher list. For the present sample a mixture of normal distributions provides a relatively reasonable approximation, although the bump around 175 would be ignored with high probability when fitting such a model. Since according to expert knowledge and analyzes of other data sets there is no 'typical shape' for such data, it is better to use a nonparametric approach which avoids specific assumptions on the shape.

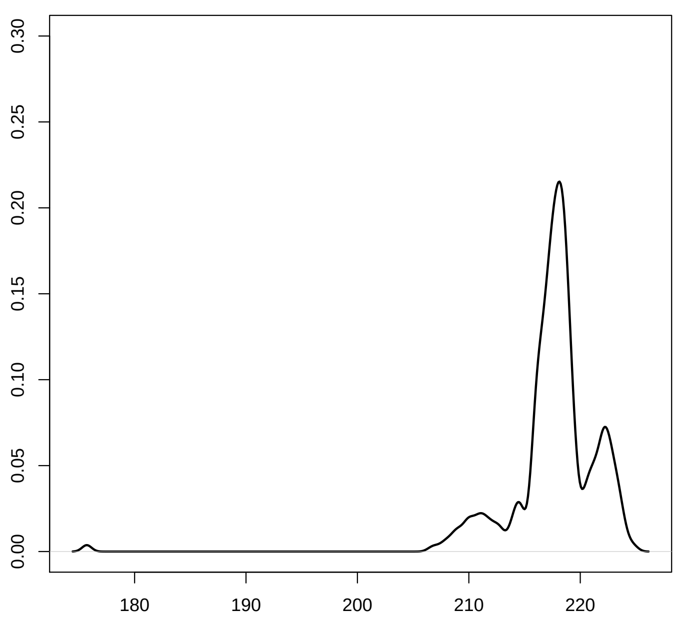

FIGURE 1.— Kernel density estimate of the flasher list measurements.

Applying our procedure with nominal error rates $1-\alpha=\beta=0.05$, AQL $p_\alpha=0.01$ and RQL $p_\beta=0.05$, we obtained $n=29$ and $c=10.82$. Thus, a random sample of 29 modules was drawn and analyzed yielding $\bar{X}'_{29}=216.30$. The decision rule was applied with $\tau=\mu_0-0.1\cdot\mu_0$, where $\mu_0=220$ specifies the nominal power output stated by the producer. Since $T_{29}=9.452$, the lot is rejected.

4.2. *Small sample properties*

To get some insight into the statistical properties, particularly the dispersion and the distributional shape of the estimates n_m and c_m, we performed Monte Carlo experiments. Having in mind the situation of the data analysis above, we selected simulation models with realistic means and variances, although, of course, the estimates n_m and c_m do not depend on location and scale. Model 1 assumes that the measurements X_i are normally distributed with mean 220 and variance 4. The other two models assume mixture distributions. Under model 2,

$$X_i \sim F_2 = 0.1N(210,6)+0.9N(230,4),$$

whereas under model 3

$$X_i \sim F_3 = 0.9N(220,4)+0.1N(230,8).$$

Note that F_2 is skewed to the left and F_3 is skewed to the right. Further, $F_2^{-1}(p_\beta)-F_2^{-1}(p_\alpha)$ is larger than $F_3^{-1}(p_\beta)-F_3^{-1}(p_\alpha)$. Thus, we expect smaller sample sizes under model 2 than under model 3.

To investigate the finite sample behavior under the above models we considered $\mathbb{E}(n_m)$, $\mathrm{sd}(n_m)$, the quartiles $q_{0.25}, q_{0.5}$, $q_{0.75}$ of the distribution of n_m, $\mathbb{E}(c_m)$, and $\mathrm{sd}(c_m)$. The sample size m of the additional data was chosen as $m\in\{100,250,500,50\,000\}$. AQL and RQL were fixed at $p_\alpha=0.02$ and $p_\beta=0.05$, respectively, and the nominal error probabilities were chosen as $\beta=1-\alpha=0.05$. The following table provides Monte-Carlo estimates based on $50\,000$ repetitions.

Model	m	$\mathbb{E}(n_m)$	$\mathrm{sd}(n_m)$	$q_{0.25}$	$q_{0.5}$	$q_{0.75}$	$\mathbb{E}(c_m)$	$\mathrm{sd}(c_m)$
1	100	299.35	17.3	51	106	239	23.13	18.86
1	250	108.73	10.43	50	79	129	17.47	6.56
1	500	83.72	9.15	52	72	101	16.11	4.05
1	50 000	65.39	8.09	63	65	68	14.89	0.35
2	100	88.57	9.41	12	28	66	17.85	16.55
2	250	35.35	5.95	16	25	42	14.74	6.08
2	500	28.65	5.35	17	24	35	14.06	3.82
2	50 000	23.45	4.84	23	23	24	13.37	0.34
3	100	614.14	24.78	86	178	414	25.37	25.52
3	250	192.55	13.88	86	137	227	19.07	7.37
3	500	147.07	12.13	90	125	178	17.57	4.45
3	50 000	114.09	10.68	110	114	118	16.19	0.38

The quantiles indicate that the distribution of n_m is quite skewed for small to moderate sample sizes m, and, depending on the true distribution, its dispersion can be quite large. Although still a bit unpleasant, for larger sample sizes of the flasher list (as, e.g., $m = 500$) the skewness is no longer a severe problem, since it implies conservative sample sizes.

The question arises how the above findings depend on the chosen values for AQL and RQL, i.e., on the parameter α. Figure 2 illustrates for model 3 in terms of the 10% and 90% quantiles how the distribution of n_{500} depends on $\alpha = 1 - \beta \in [0.9, 0.99]$.

To summarize, the above simulations support the applicability of our proposal and the assertion of the asymptotic results, although the method is not practical in all cases. An investigation of, e.g., a modified rule or a procedure employing accompanying rules is beyond the scope of the present article.

4.3. *Investigation of the moment condition*

The above results provide some insight into the behavior of our procedure for small to moderate sample sizes, if the assumptions on the underlying distribution are satisfied. But what does happen if they are not satisfied? To shed some light on that issue we investigated to which extent the fourth moment condition of Assumption 2.1 is really required. For that purpose we simulated measurements following a symmetric Pareto type distribution, i.e.,

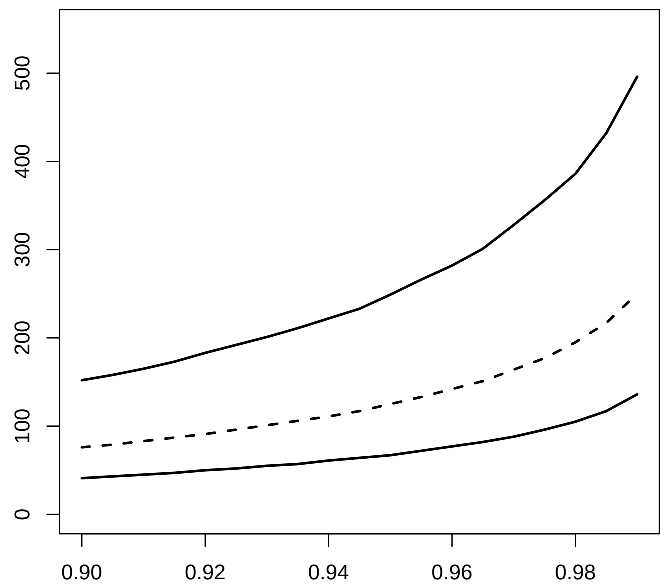

FIGURE 2.— Median (dashed line), 10%-, and 90%-quantiles of the distribution of n_m as a function of $\alpha = 1 - \beta$.

for $\gamma > 0$ we put

$$G_\gamma(x) = \begin{cases} 1 - \frac{1}{2}(1+x)^{-\gamma}, & x \geq 0, \\ \frac{1}{2}(1-x)^{-\gamma}, & x < 0, \end{cases}$$

to define the location-scale family $F_{(\mu,\sigma^2),\gamma}(x) = G_\gamma((x-\mu)/\sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$. As is well known, the rth moment of G_γ exists for all $0 < r < \gamma$, but the γth moment does not. Since we are interested now on the large sample (asymptotic) distribution, we put $m = 20\,000$. As in our first analysis, AQL and RQL were chosen as $p_\alpha = 0.02$ and $p_\beta = 0.05$, respectively, and the nominal error probabilities are given by $\beta = 1 - \alpha = 0.05$. For fixed $\gamma > 0$ the distance between the empirical distribution function of the standardized simulated n_m values, denoted by $\widehat{H}_\gamma$, and the $N(0,1)$ distribution function was measured by the Kolmogorov-Smirnov statistic. This means, for S simulated values $n_m(1), \ldots, n_m(S)$, we calculated

$$\widehat{D}_\gamma = \sup_x |\widehat{H}_\gamma(x) - \Phi(x)|.$$

where

$$\widehat{H}_\gamma(x) = \frac{1}{S}\sum_{i=1}^{S} \mathbf{1}_{(-\infty,x]}((n_m(i) - \overline{n}_m)/s_m), \qquad x \in \mathbb{R},$$

with $\overline{n}_m = S^{-1}\sum_{i=1}^S n_m(i)$ and $s_m^2 = (S-1)^{-1}\sum_{i=1}^S (n_m(i) - \overline{n}_m)^2$. Notice that $\widehat{D}_\gamma$ consistently estimates $D_\gamma = \|H_\gamma - \Phi\|_\infty$, where H_γ stands for the true distribution of $(n_m - \mathbb{E}(n_m))/(\mathbb{V}(n_m))^{1/2}$. For each γ the Monte Carlo estimate $\widehat{D}_\gamma$ was based on $S =$ 500 000 i.i.d. replications, independent from each other.

If the existence of the fourth moment is required for the theory to hold, $\widehat{D}_\gamma$ should be small for $\gamma > 4$ and get large as γ decreases. Figure 3 depicts $\widehat{D}_\gamma$ for $\gamma = 2.0, 2.1, \ldots, 4.5$. Since there is still some estimation error and to support visual evaluation, we added a Nadaraya-Watson kernel smooth of the simulated values using a bandwidth 0.5. The result indicates that there is no hope to weaken Assumption 2.1.

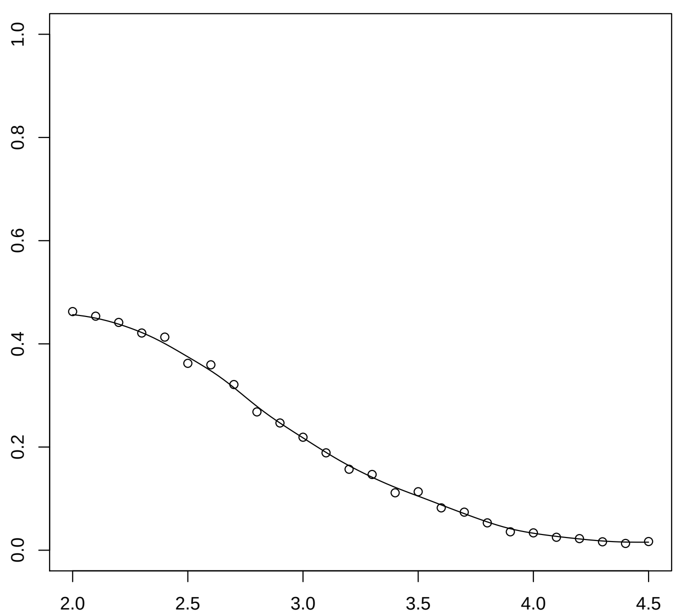

FIGURE 3.— Kolmogorov-Smirnov distance, $\widehat{D}_\gamma$, between the empirical distribution of n_m and the d.f. of the standard normal distribution under a Pareto type distribution, G_γ, for $\gamma = 2.0, 2.1, \ldots, 4.5$.

5. PROOFS OF MAIN RESULTS

The crux of the proofs of Theorems 3.1 and 3.2 (which will be carried out in Subsections 5.1 and 5.2, respectively) is Lemma 5.2 below. The proof of this lemma relies on the functional delta method (which is recalled in the Appendix C), and on Lemma 5.1 (resp. Corollary 5.1) for which we need some notation. We write $D(\bar{\mathbb{R}})$ for the space of all cádlàg functions on $\bar{\mathbb{R}}$. Moreover, we adopt Dudley's notion of convergence in distribution of random elements in $D(\bar{\mathbb{R}})$ with respect to the uniform metric (see, for instance, [9, Chapter V]). In particular, we regard $D(\bar{\mathbb{R}})$ as a measurable space with respect to the σ-algebra generated by all finite-dimensional projections (or, equivalently, by the $\|.\|_\infty$-closed balls). We also equip $D(\bar{\mathbb{R}})$ with the supremum norm $\|.\|_\infty$ to make it a normed space. For the considerations in this section we may assume without loss of generality $\Delta = 0$ in (2.10) and (2.11). In particular, we may and do write for the sake of clarity X_i instead of X_i^Δ. We refer to the transpose of an Euclidean vector v as v', and we set, for $t \in \mathbb{R}$,

$$F_m(t) := \frac{1}{m}\sum_{i=1}^{m} \mathbf{1}_{(-\infty,t]}(X_i) \qquad \text{and} \qquad \hat{F}_m(t) := G\left(\frac{t-\bar{X}_m}{\sigma_m}\right).$$

To simplify the exposition we will sometimes refer to the distribution function $F_{(\mu,\sigma^2)}$ of the X_i under $\mathbb{P}_{(\mu,\sigma^2)}$ simply as F.

LEMMA 5.1 *Suppose Assumptions 2.1-2.2 hold. Then, under $\mathbb{P}_{(\mu,\sigma^2)}$, as $m \to \infty$,*

$$\sqrt{m}\left(F_m - \hat{F}_m\right) \xrightarrow{d} \hat{B}_F^\circ \qquad (\text{in } D(\bar{\mathbb{R}})). \tag{5.1}$$

Here $\hat{B}_F^\circ(.) \overset{d}{=} B_F^\circ(.) - [\dot{F}_{(\mu,\sigma^2)}(.)](\xi_\mu, \xi_{\sigma^2})'$, where $\dot{F}_{(\mu,\sigma^2)}$ is defined as in (2.2), B_F° is an F-Brownian bridge, and ξ_μ, ξ_{σ^2} and B_F° are jointly normal with 0 mean and

$$\mathbb{C}\mathrm{ov}_{(\mu,\sigma^2)}(\xi_\theta, \xi_{\theta'}) = \mathbb{C}\mathrm{ov}_{(\mu,\sigma^2)}\left(\psi_\theta(X_1), \psi_{\theta'}(X_1)\right)$$

$$\mathbb{C}\mathrm{ov}_{(\mu,\sigma^2)}(\xi_\theta, B_F^\circ(t)) = \mathbb{C}\mathrm{ov}_{(\mu,\sigma^2)}\left(\psi_\theta(X_1), \mathbf{1}_{(-\infty,t]}(X_1)\right)$$

for $\theta, \theta' \in \{\mu, \sigma^2\}$, $\psi_\mu(x) := x - \mu$ and $\psi_{\sigma^2}(x) := (x-\mu)^2 - \sigma^2$.

On an informal level, $\hat{B}_F^\circ$ can be seen as an "F-Brownian bridge with drift". Note that this process is continuous since B_F° and $[\dot{F}_{(\mu,\sigma^2)}(.)](\xi_\mu, \xi_{\sigma^2})'$ are; recall that F and $\dot{F}_{(\mu,\sigma^2)}$ are continuous. In particular, it actually does not matter whether we consider convergence in distribution w.r.t. the uniform metric or w.r.t. the Skorohod metric (cf. [1, p.110]). We imposed the supremum metric for the simple reason that we intend to apply the functional delta method later on.

REMARK 5.1 *It is straightforward to check that the covariance function, $\Gamma_{(\mu,\sigma^2)}$, of the Gaussian process $\hat{B}_F^\circ$ in Lemma 5.1 is given by*

$$\begin{aligned}\Gamma_{(\mu,\sigma^2)}(s,t) &= F_{(\mu,\sigma^2)}(s)\left(1-F_{(\mu,\sigma^2)}(t)\right) - \dot{F}_{(\mu,\sigma^2)}(t)b_{(\mu,\sigma^2)}(s) \\ &\quad -\dot{F}_{(\mu,\sigma^2)}(s)b_{(\mu,\sigma^2)}(t) + \dot{F}_{(\mu,\sigma^2)}(s)A_{(\mu,\sigma^2)}\dot{F}_{(\mu,\sigma^2)}(t)'\end{aligned} \tag{5.2}$$

for $s \le t$, where $\dot{F}_{(\mu,\sigma^2)}$ is defined as in (2.2), and

$$b_{(\mu,\sigma^2)}(t) := \begin{pmatrix} \mathbb{E}_{(\mu,\sigma^2)}\left[(X_1-\mu)\mathbf{1}_{(-\infty,t]}(X_1)\right] \\ \mathbb{E}_{(\mu,\sigma^2)}\left[\left((X_1-\mu)^2-\sigma^2\right)\mathbf{1}_{(-\infty,t]}(X_1)\right] \end{pmatrix},$$

$$A_{(\mu,\sigma^2)} := \begin{pmatrix} \sigma^2 & \mathbb{E}_{(\mu,\sigma^2)}[(X_1-\mu)^3] \\ \mathbb{E}_{(\mu,\sigma^2)}[(X_1-\mu)^3] & \mathbb{E}_{(\mu,\sigma^2)}[(X_1-\mu)^4]-\sigma^4 \end{pmatrix}.$$

PROOF: (of Lemma 5.1) The proof of Lemma 5.1 is an application of a result which is mainly associated with Darling and Durbin, cf. [9, Example V.15], or [17, Theorem 19.23]. According to that, it suffices to show that the mapping $(\mu,\sigma^2) \mapsto F_{(\mu,\sigma^2)}$ is Fréchet differentiable at (μ,σ^2), and that

$$\sqrt{m}\begin{pmatrix} \bar{X}_m-\mu \\ \sigma_m^2-\sigma^2 \end{pmatrix} = \frac{1}{\sqrt{m}}\sum_{i=1}^m \begin{pmatrix} \psi_\mu(X_i) \\ \psi_{\sigma^2}(X_i) \end{pmatrix} + o_{\mathbb{P}_{(\mu,\sigma^2)}}(1) \tag{5.3}$$

holds for some functions ψ_μ and ψ_{σ^2} satisfying $\mathbb{E}_{(\mu,\sigma^2)}[\psi_\mu(X_1)] = 0$, $\mathbb{E}_{(\mu,\sigma^2)}[\psi_{\sigma^2}(X_1)] = 0$ and $\mathbb{E}_{(\mu,\sigma^2)}\left[\sum_{\theta\in\{\mu,\sigma^2\}}\psi_\theta^2(X_1)\right] < \infty$. Since the fourth moment of G is assumed to be finite (Assumption 2.1), we may apply the central limit theorem to the random variables $(X_i-\mu)^2$, $i=1,2,\ldots$. In view of this, it is easy to see that (5.3) is fulfilled for ψ_μ and ψ_{σ^2} defined as in the statement of Lemma 5.1. The Fréchet differentiability is ensured by Assumption 2.2. *Q.E.D.*

COROLLARY 5.1 *Suppose Assumptions 2.1-2.2 hold, and set $\eta(t) := \sigma t + \mu$. Then, under $\mathbb{P}_{(\mu,\sigma^2)}$, as $m \to \infty$,*

$$\sqrt{m}\left(G_m - G\right) \xrightarrow{d} \hat{B}_F^\circ \circ \eta \overset{d}{=} \hat{B}_G^\circ \qquad (\textit{in } D(\bar{\mathbb{R}})), \tag{5.4}$$

*where $\hat{B}_F^\circ$ is as in Lemma 5.1. $\hat{B}_G^\circ$ is a centered Gaussian processes with covariance function given by (5.2) with $(\mu,\sigma^2)=(0,1)$, i.e., by $\Gamma_{(0,1)} = \Gamma$ from (**??**).*

PROOF: We denote by $\bar{C}(\bar{\mathbb{R}})$ the space of strictly increasing $\bar{\mathbb{R}}$-valued continuous functions ψ on $\bar{\mathbb{R}}$ with $\psi(-\infty) = -\infty$ and $\psi(+\infty) = +\infty$. We equip $\bar{C}(\bar{\mathbb{R}})$ with the Borel-σ-algebra

$\bar{\mathcal{C}}$ related to the metric $d_\infty(\psi_1,\psi_2) := \sum_{k=1}^{\infty} 2^{-k}\min\{1,\sup_{x\in[-k,k]}|\psi_1(x)-\psi_2(x)|\}$. The process $\eta_m(t) := \sigma_m t + \bar{X}_m$ clearly converges in probability, and therefore in distribution, (in $\bar{C}(\bar{\mathbb{R}})$) to the deterministic process η. Since $\hat{B}_F^\circ$ is continuous, the convergence in (5.1) holds in particular w.r.t. the Borel-σ-algebra, $\mathcal{D}$, related to the Skorohod metric. Hence we may apply the theory of [5]. Firstly, Theorem 4.4 of [5] yields convergence in distribution of $(\sqrt{m}\,(F_m-\hat{F}_m),\eta_m)$ to $(\hat{B}_F^\circ,\eta)$ on $\mathcal{D}\times\bar{\mathcal{C}}$. Note that the product σ-algebra $\mathcal{D}\times\bar{\mathcal{C}}$ coincides with the Borel-σ-algebra related to $d((\phi_1,\psi_1),(\phi_2,\psi_2)) := \max\{\|\phi_1-\phi_2\|_\infty, d_\infty(\psi_1,\psi_2)\}$. The map $(\phi,\psi)\mapsto\phi\circ\psi$ from $D(\bar{\mathbb{R}})\times\bar{C}(\bar{\mathbb{R}})$ to $D(\bar{\mathbb{R}})$ is easily seen to be $(d,\|.\|_\infty)$-continuous at each point of $C(\bar{\mathbb{R}})\times\bar{C}(\bar{\mathbb{R}})$ (we write $C(\bar{\mathbb{R}})$ for the space of continuous functions on $\bar{\mathbb{R}}$). Since moreover $(\hat{B}_F^\circ,\eta)$ is concentrated on $C(\bar{\mathbb{R}})\times\bar{C}(\bar{\mathbb{R}})$, Corollary 1 to Theorem 5.1 of [5] shows that $\sqrt{m}\,(F_m-\hat{F}_m)\circ\eta_m \xrightarrow{d} \hat{B}_F^\circ\circ\eta$ on $\mathcal{D}$. The limit $\hat{B}_F^\circ\circ\eta$ is of course continuous, and therefore the convergence in distribution holds w.r.t. the uniform metric too. But this proves the convergence in distribution to $\hat{B}_F^\circ\circ\eta$ in (5.4) since $G_m = F_m\circ\eta_m$ and $G = \hat{F}_m\circ\eta_m$. It remains to show $\hat{B}_F^\circ\circ\eta \stackrel{d}{=} \hat{B}_G^\circ$. For $s\le t$ we obviously have

$$\mathbb{C}\mathrm{ov}_{(\mu,\sigma^2)}(\hat{B}_G^\circ(s),\hat{B}_G^\circ(t)) = \mathbb{C}\mathrm{ov}_{(\mu,\sigma^2)}(\hat{B}_F^\circ\circ\eta(s),\hat{B}_F^\circ\circ\eta(t)) = \Gamma_{(\mu,\sigma^2)}(\eta(s),\eta(t)).$$

Then straightforward calculations, using $\mathbb{E}_{(\mu,\sigma^2)}[(X_1-\mu)^k] = \sigma^k\mathbb{E}_{(0,1)}[X_1^k]$, show that this expression coincides with $\Gamma_{(0,1)}(s,t)$. *Q.E.D.*

Lemma 5.2 *Suppose Assumptions 2.1-2.3 hold. Then, for every $k\in\mathbb{N}$, $p_1,\ldots,p_k\in(0,1)$ and $\lambda_1,\ldots,\lambda_k\in\mathbb{R}$,*

$$\sqrt{m}\;\sum_{i=1}^{k}\lambda_i\left(G_m^{-1}(p_i)-G^{-1}(p_i)\right)\;\xrightarrow{d}\;\sum_{i=1}^{k}\lambda_i\,\frac{\hat{B}_G^\circ(G^{-1}(p_i))}{G'(G^{-1}(p_i))}\qquad(\textit{in }\mathbb{R})$$

under $\mathbb{P}_{(\mu,\sigma^2)}$, as $m\to\infty$, where $\hat{B}_G^\circ$ is as in Corollary 5.1.

Proof: Let $\mathbb{D}$ be the subset of $D(\bar{\mathbb{R}})$ that consists of all non-decreasing functions ϕ satisfying $\phi(-\infty)=0$ and $\phi(+\infty)=1$. Moreover, let $\mathbb{D}_0$ be the subset of $D(\bar{\mathbb{R}})$ that consists of all functions ϕ being continuous at $\phi^{-1}(p)$, and let $Q_p:\mathbb{D}\to\mathbb{R}$ be defined by $Q_p(\phi) := \phi^{-1}(p)$. Because of $0<G'(G^{-1}(p))<\infty$ (by Assumption 2.3) and the continuity of G, it can be shown (with the help of the Arzelà-Ascoli theorem) that Q_p is Hadamard differentiable at G tangentially to $\mathbb{D}_0$ with Hadamard derivative

$$D_{G;\mathbb{D}_0}^{Had}Q_p\;(\phi) = \frac{\phi(G^{-1}(p))}{G'(G^{-1}(p))}\qquad(\phi\in\mathbb{D}_0).$$

This is more or less standard (cf. [1, Proposition II.8.4], for example), and therefore we omit the details. As an immediate consequence we obtain

$$D_{G;\mathbb{D}_0}^{Had}Q\;(\phi) = \sum_{i=1}^{k}\lambda_i\frac{\phi(G^{-1}(p_i))}{G'(G^{-1}(p_i))}\qquad(\phi\in\mathbb{D}_0)\tag{5.5}$$

for $Q : \mathbb{D} \to \mathbb{R}$ defined by $Q(\phi) := \sum_{i=1}^{k} \lambda_i Q_{p_i}(\phi)$. Because of Corollary 5.1 and (5.5), we can apply the functional Delta method (cf. the Appendix C) to obtain

$$\sqrt{m}\,\sum_{i=1}^{k}\lambda_i\left(G_m^{-1}(p_i)-G^{-1}(p_i)\right)\ \xrightarrow{d}\ D^{Had}_{G;\mathbb{D}_0}Q\ (\hat{B}_G^{\circ})\qquad(\text{in }\mathbb{R})\tag{5.6}$$

as $m \to \infty$. By virtue of (5.5), this proves Lemma 5.2. *Q.E.D.*

5.1. *Proof of Theorem 3.1*

We are now in the position to finish the proof of Theorem 3.1 without further obstacles. By (1.1) we have $p = \mathbb{P}_{(\mu,\sigma^2)}[X_1 \le \tau]$, and therefore

$$\frac{\tau-\mu}{\sigma} = G^{-1}(p).\tag{5.7}$$

Now, $T_n > c$ holds if and only if

$$\sqrt{n}\ \frac{\bar{X}_n'-\mu}{\sigma}+\sqrt{n}\Big(\frac{\bar{X}_n'-\mu}{\sigma_m}-\frac{\bar{X}_n'-\mu}{\sigma}\Big)\ >\ c+\sqrt{n}\ G_m^{-1}(p)+\sqrt{n}\Big(\frac{\tau-\mu}{\sigma_m}-G_m^{-1}(p)\Big)$$

which in turn is equivalent to

$$\begin{aligned}&\sqrt{n}\ \frac{\bar{X}_n'-\mu}{\sigma}+\sqrt{n}\ \frac{\bar{X}_n'-\mu}{\sigma}\ \frac{\sigma-\sigma_m}{\sigma_m}\\ &\quad+\sqrt{n}\Big(G_m^{-1}(p)-G^{-1}(p)\Big)+\sqrt{n}\Big(G^{-1}(p)-\frac{\tau-\mu}{\sigma_m}\Big)\ >\ c+\sqrt{n}\ G^{-1}(p).\end{aligned}\tag{5.8}$$

We denote the four summands on the left-hand side of (5.8) by $S_1(n), \ldots, S_4(n)$. The first one is asymptotically standard normal by the central limit theorem. Because $\frac{\sigma-\sigma_m}{\sigma_m}$ converges $\mathbb{P}_{(\mu,\sigma^2)}$-almost surely to $\frac{\sigma-\sigma}{\sigma} = 0$ by the strong law of large numbers, $S_2(n)$ converges in probability to 0 by Slutsky's lemma. According to Lemma 5.2 and (2.12), $S_3(n)$ also converges in probability to 0 by Slutsky's lemma. We further obtain by (5.7),

$$S_4(n)=\sqrt{n}\left(G^{-1}(p)-G^{-1}(p)\frac{\sigma}{\sigma_m}\right)=\frac{G^{-1}(p)}{\sigma_m}\ \frac{\sqrt{n}}{\sqrt{m}}\ \sqrt{m}\,(\sigma_m-\sigma).$$

Now, $\sqrt{m}(\sigma_m - \sigma)$ can be split into the sum of $U_1(m) := \sqrt{m}(\sigma_m - \tilde{\sigma}_m)$ and $U_2(m) := \sqrt{m}(\tilde{\sigma}_m - \sigma)$, where $\tilde{\sigma}_m := (\frac{1}{m}\sum_{i=1}^{m}(X_i - \mu)^2)^{1/2}$. The summand $U_2(m)$ is asymptotically normal with 0 mean and variance $\mathbb{V}\mathrm{ar}_{(\mu,\sigma^2)}[(X_1 - \mu)^2]$. Multiplying by $\sqrt{n}/\sqrt{m}$ yields convergence in probability to 0, since we assumed (2.12). For $m \geq 2$ we also obtain

$$\begin{aligned} |U_1(m)| &= \sqrt{m}\left|\left(\frac{1}{m-1}\sum_{i=1}^m (X_i - \bar{X}_m)^2\right)^{1/2} - \left(\frac{1}{m-1}\sum_{i=1}^m (X_i-\mu)^2\right)^{1/2}\right| \\ &\leq \sqrt{m}\frac{\sqrt{m}}{\sqrt{m-1}}\left|\frac{1}{m}\sum_{i=1}^m (X_i-\bar{X}_m)^2 - \frac{1}{m}\sum_{i=1}^m (X_i-\mu)^2\right|^{1/2} \\ &\leq \sqrt{m}\,\sqrt{2}\left|\frac{1}{m}\sum_{i=1}^m \Big(-\bar{X}_m^2 + 2X_i\mu - \mu^2\Big)\right|^{1/2} \\ &= \sqrt{2}\,\sqrt{m}\,\left|-\bar{X}_m^2 + 2\bar{X}_m\mu - \mu^2\right|^{1/2} \\ &= \sqrt{2}\,\sqrt{m}\,\left|\bar{X}_m - \mu\right|. \end{aligned}$$

Thus, since we assumed (2.12) and since $\sqrt{m}(\bar{X}_m - \mu)$ is asymptotically $N(0,\sigma^2)$, we deduce with the help of Slutsky's lemma that $(\sqrt{n}/\sqrt{m})|U_1(m)|$ converges in probability to 0. Moreover, $G^{-1}(p)/\sigma_m$ converges almost surely to $G^{-1}(p)/\sigma$, so that Slutsky's lemma ensures that $S_4(n)$ converges in probability to 0. This completes the proof of Theorem 3.1.

5.2. *Proof of Theorem 3.2*

The proof of Theorem 3.2 relies on the delta method for random variables (cf. the Appendix C) and Lemma 5.2. Indeed, Lemma 5.2 implies

$$\sqrt{m}\Big(G_m^{-1}(p_\beta) - G_m^{-1}(p_\alpha) - (G^{-1}(p_\beta) - G^{-1}(p_\alpha))\Big) \xrightarrow{d} \frac{\hat{B}_G^\circ(G^{-1}(p_\beta))}{G'(G^{-1}(p_\beta))} - \frac{\hat{B}_G^\circ(G^{-1}(p_\alpha))}{G'(G^{-1}(p_\alpha))}.$$

Now, the left-hand side of (3.8) can be expressed as

$$\sqrt{m}\Big(f(G_m^{-1}(p_\beta) - G_m^{-1}(p_\alpha)) - f(G^{-1}(p_\beta) - G^{-1}(p_\alpha))\Big)$$

with $f(x) = (\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta))^2\, x^{-2}$, recall (3.6). Therefore, Theorem 3.1 of [17] shows that the left-hand side of (3.8) converges weakly to the law of the normal random variable

$$-2\,\frac{(\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta))^2}{(G^{-1}(p_\beta) - G^{-1}(p_\alpha))^3}\left(\frac{\hat{B}_G^\circ(G^{-1}(p_\beta))}{G'(G^{-1}(p_\beta))} - \frac{\hat{B}_G^\circ(G^{-1}(p_\alpha))}{G'(G^{-1}(p_\alpha))}\right)$$

since $f'(x) = -2\,(\Phi^{-1}(1-\alpha) - \Phi^{-1}(1-\beta))^2\, x^{-3}$. Now the proof can be completed easily since we know the covariance function, $\Gamma_{(0,1)} = \Gamma$, of $\hat{B}_G^\circ$ from Corollary 5.1.

APPENDIX A: PROOF OF (2.3)

This appendix is devoted to a proof of the sufficiency of condition (2.3) (along with smoothness of G) for (2.1). This result is quite interesting on its own. A similar analysis in a more general setting can be found in [15].

Lemma A.1 *If G is twice continuously differentiable and satisfies (2.3), then (2.1) holds.*

Proof: Let $f_{(\mu,\sigma^2)}$ and g denote the Lebesgue densities of $F_{(\mu,\sigma^2)}$ and G, respectively. Since g is continuously differentiable and

$$f_{(\mu,\sigma^2)}(x) = \frac{1}{\sigma}\, g\Big(\frac{x-\mu}{\sigma}\Big), \tag{A.1}$$

we obtain continuity of the mapping $(\sigma^2, x) \mapsto \frac{\partial}{\partial \sigma^2} f_{(\mu,\sigma^2)}(x)$ restricted to $\sigma^2 > 0$. Together with the integrability of $f_{(\mu,\sigma^2)}$, this justifies the following interchange of the derivative and the integral,

$$\frac{\partial}{\partial \sigma^2} F_{(\mu,\sigma^2)}(t) = \frac{\partial}{\partial \sigma^2} \int_{-\infty}^{t} f_{(\mu,\sigma^2)}(x)dx = \int_{-\infty}^{t} \frac{\partial}{\partial \sigma^2} f_{(\mu,\sigma^2)}(x)dx.$$

In the same way we obtain the analogue for "$\frac{\partial}{\partial \mu}$". With the help of (A.1) we deduce for $\dot F_{(\mu,\sigma^2)}$, defined in (2.2),

$$\dot F_{(\mu,\sigma^2)}(t) = \bigg(-\frac{1}{\sigma^2}\int_{-\infty}^{t} g'\Big(\frac{x-\mu}{\sigma}\Big)dx,\ -\frac{1}{\sigma^2}\int_{-\infty}^{t} \frac{x-\mu}{2\sigma^2}\, g'\Big(\frac{x-\mu}{\sigma}\Big)dx \bigg).$$

Thus, (2.1) follows if we can prove that

$$\int_{-\infty}^{t} \frac{1}{|v|}\bigg|\frac{1}{\sigma}\, g\Big(\frac{x-(\mu+v_1)}{\sqrt{\sigma^2+v_2}}\Big) - \frac{1}{\sigma}\, g\Big(\frac{x-\mu}{\sigma}\Big) - \frac{1}{\sigma^2}\,\Big(v_1 + v_2\frac{x-\mu}{2\sigma^2}\Big) g'\Big(\frac{x-\mu}{\sigma}\Big)\bigg| dx \to 0 \tag{A.2}$$

uniformly in t, as $|v| \to 0$, where $v = (v_1, v_2)'$. In the remainder of the proof we will establish (A.2).

Let $\epsilon > 0$. We split the integral in (A.2) into

$$\int_{(-\infty,t]} \cdots = \int_{(-\infty,t]\cap[a_\epsilon,b_\epsilon]^c} \cdots + \int_{(-\infty,t]\cap[a_\epsilon,b_\epsilon]} \cdots \ =: \ S_1^\epsilon(t,v) + S_2^\epsilon(t,v)$$

for some real numbers a_ϵ, b_ϵ satisfying $a_\epsilon < b_\epsilon$. We intend to prove (A.2) by showing that both $S_1^\epsilon(t,v)$ and $S_1^\epsilon(t,v)$ are bounded by $\epsilon/2$ uniformly in t for sufficiently small $|v|$, for a suitable choice of a_ϵ and b_ϵ. To estimate $S_1^\epsilon(t,v)$ we note that, by the Mean Value Theorem, we have for some ξ inbetween $(x-\mu)/\sigma$ and $(x-(\mu+v_1))/\sqrt{\sigma^2+v_2}$,

$$\frac{1}{\sigma}\, g\Big(\frac{x-(\mu+v_1)}{\sqrt{\sigma^2+v_2}}\Big) - \frac{1}{\sigma}\, g\Big(\frac{x-\mu}{\sigma}\Big) = \Big(\frac{x-(\mu+v_1)}{\sqrt{\sigma^2+v_2}} - \frac{x-\mu}{\sigma}\Big)\frac{1}{\sigma}\, g'\Big(\xi\Big). \tag{A.3}$$

Hence the integrand in (A.2) is bounded by

$$\frac{\left|\frac{x-(\mu+v_1)}{\sqrt{\sigma^2+v_2}}-\frac{x-\mu}{\sigma}\right|}{\sqrt{v_1^2+v_2^2}}\left|\frac{1}{\sigma}g'(\xi)\right| + \frac{\left|v_1+v_2\frac{x-\mu}{2\sigma^2}\right|}{\sqrt{v_1^2+v_2^2}}\left|\frac{1}{\sigma^2}g'\Big(\frac{x-\mu}{\sigma}\Big)\right|$$

$$\le \left|\frac{x\sigma-(\mu+v_1)\sigma-x\sqrt{\sigma^2+v_2}+\mu\sqrt{\sigma^2+v_2}}{\sigma\sqrt{\sigma^2+v_2}\sqrt{v_1^2+v_2^2}}\right|\left|\frac{1}{\sigma}g'(\xi)\right| + \left|1+\frac{x-\mu}{2\sigma^2}\right|\left|\frac{1}{\sigma^2}g'\Big(\frac{x-\mu}{\sigma}\Big)\right|$$

$$= \frac{1}{\sigma^2}\frac{\left|(x-\mu)(\sigma-\sqrt{\sigma^2+v_2})-v_1\sigma\right|}{\sqrt{\sigma^2+v_2}\sqrt{v_1^2+v_2^2}}\,|g'(\xi)| + \frac{1}{\sigma^2}\left|1+\frac{x-\mu}{2\sigma^2}\right|\left|g'\Big(\frac{x-\mu}{\sigma}\Big)\right|$$

$$\le \frac{1}{\sigma^2}\frac{|x-\mu|\cdot|v_2|\frac{1}{2\sqrt{\min\{\sigma^2,(\sigma^2+v_2)\}}}+|v_1|\sigma}{\sqrt{\sigma^2+v_2}\sqrt{v_1^2+v_2^2}}\,|g'(\xi)| + \frac{1}{\sigma^2}\left|1+\frac{x-\mu}{2\sigma^2}\right|\left|g'\Big(\frac{x-\mu}{\sigma}\Big)\right|.$$

For the latter step we used the inequality

$$\text{(A.4)}\qquad |\sqrt{a}-\sqrt{b}| = \left|\frac{a-b}{\sqrt{a}+\sqrt{b}}\right| \le \frac{|a-b|}{2\sqrt{\min\{a,b\}}} \qquad (a,b\ge 0)$$

to estimate the expression $|\sigma-\sqrt{\sigma^2+v_2}| = |\sqrt{\sigma^2}-\sqrt{\sigma^2+v_2}|$. Now it follows easily that the integrand in (A.2) is bounded by

$$C_{\mu,\sigma}\,(1+|x|)\left(|g'(\xi)| + \left|g'\Big(\frac{x-\mu}{\sigma}\Big)\right|\right)$$

for some constant $C_{\mu,\sigma}>0$ that may depend on μ and σ, provided $|v|$ is so small so that (σ^2+v_2) is larger than (for instance) $\sigma^2/2$. Since $g'=G''$, we deduce with the help of our basic assumption (2.3) that the integral in (A.2) with $t=\infty$ is finite. Because of this and the continuity of the integrand, we may choose a_ϵ and b_ϵ in such a way that $S_1^\epsilon(t,v)<\epsilon/2$ for all $t\in\mathbb{R}$ (and sufficiently small $|v|$).

It remains to show $S_2^\epsilon(t,v)<\epsilon/2$ for all $t\in\mathbb{R}$ (and sufficiently small $|v|$). With the help of (A.3) we obtain as before that the integrand in (A.2) equals

$$\left|\frac{\frac{x-(\mu+v_1)}{\sqrt{\sigma^2+v_2}}-\frac{x-\mu}{\sigma}}{\sqrt{v_1^2+v_2^2}}\,\frac{1}{\sigma}g'(\xi) - \frac{\left(v_1+v_2\frac{x-\mu}{2\sigma^2}\right)}{\sqrt{v_1^2+v_2^2}}\,\frac{1}{\sigma^2}g'\Big(\frac{x-\mu}{\sigma}\Big)\right|.$$

By the triangle inequality this expression is bounded above by

$$\left|\frac{\frac{x-(\mu+v_1)}{\sqrt{\sigma^2+v_2}}-\frac{x-\mu}{\sigma}}{\sqrt{v_1^2+v_2^2}} - \frac{\left(v_1+v_2\frac{x-\mu}{2\sigma^2}\right)}{\sqrt{v_1^2+v_2^2}}\,\frac{1}{\sigma^2}g'\big(\xi\big)\right|$$

$$+\left|\frac{\left(v_1+v_2\frac{x-\mu}{2\sigma^2}\right)}{\sqrt{v_1^2+v_2^2}}\,\frac{1}{\sigma^2}g'\big(\xi\big) - \frac{\left(v_1+v_2\frac{x-\mu}{2\sigma^2}\right)}{\sqrt{v_1^2+v_2^2}}\,\frac{1}{\sigma^2}g'\big(\xi\big)\right|.$$

Doing some elementary manipulations, we obtain the upper bound

$$\begin{aligned}
\text{(A.5)}\quad & \left(|x-\mu| \left| \frac{\frac{\sqrt{\sigma^2+v_2}-\sqrt{\sigma^2}}{\sqrt{\sigma^2+v_2}} - \frac{v_2}{2\sigma^2}}{\sqrt{v_1^2+v_2^2}} \right| + \left| \frac{v_1\left(\frac{\sigma^2}{\sqrt{\sigma^2+v_2}}-1\right)}{\sqrt{v_1^2+v_2^2}} \right| \right) \frac{1}{\sigma} g'(\xi) \\
& \qquad + \left| \frac{\left(v_1 + v_2 \frac{x-\mu}{2\sigma^2}\right)}{\sqrt{v_1^2+v_2^2}} \right| \frac{1}{\sigma^2} \left| g'\left(\frac{x-\mu}{\sigma}\right) - g'(\xi) \right| \\
=:\; & \Big(|x-\mu| |U_1(v)| + |U_2(v)| \Big) \frac{1}{\sigma} g'(\xi) \; + \; |U_3(v)| \frac{1}{\sigma^2} \left| g'\left(\frac{x-\mu}{\sigma}\right) - g'(\xi) \right|.
\end{aligned}$$

Now,

$$\begin{aligned}
U_1(v) &= \frac{\sqrt{\sigma^2+v_2}-\sqrt{\sigma^2}}{v_2} \frac{1}{\sqrt{\sigma^2+v_2}} - \frac{1}{2\sigma^2} + o_{|v|}(1) \\
&\le \frac{v_2 \frac{1}{2\min\{\sqrt{\sigma^2,\sigma^2+v_2}\}}}{v_2} \frac{1}{\sqrt{\sigma^2+v_2}} - \frac{1}{2\sigma^2} + o_{|v|}(1),
\end{aligned}$$

where we used (A.4). Thus we have $U_1(v) \to 0$ as $|v| \to 0$. It is easy to see that also $U_2(v) \to 0$ as $|v| \to 0$. In both cases the convergence holds uniformly in t and x. Moreover, $U_3(v)$ is bounded uniformly in t and x, for sufficiently small $|v|$. As g' is continuous, it is uniformly continuous on $[a_\epsilon, b_\epsilon]$ (a_ϵ, b_ϵ have been fixed above). Therefore $|g'((x-\mu)/\sigma) - g'(\xi)|$ converges to 0 uniformly in $t \in \mathbb{R}$ and $x \in [a_\epsilon, b_\epsilon]$, as $|v| \to 0$. Altogether, the expression in (A.5) converges to 0 uniformly in $t \in \mathbb{R}$ and $x \in [a_\epsilon, b_\epsilon]$, as $|v| \to 0$. Hence $S_2^\epsilon(t,v)$ is indeed smaller than $\epsilon/2$ for all $t \in \mathbb{R}$ (and sufficiently small $|v|$). *Q.E.D.*

APPENDIX B: OPTIMAL SAMPLING PLAN

We shall now provide the details of the following fact. If (2.7) is plugged into (2.5)-(2.6), the optimal sampling plan (n,c), i.e., the sampling plan with the minimal n, satisfying (2.5)-(2.6) is given by (2.8)-(2.9). Since the right-hand side of (2.7) is strictly decreasing in p, the requirements (2.5)-(2.6) are equivalent to

$$\text{(B.1)}\quad \Phi^{-1}(1-\alpha) \;\ge\; c + \sqrt{n}\, G^{-1}(p_\alpha),$$

$$\text{(B.2)}\quad \Phi^{-1}(1-\beta) \;\le\; c + \sqrt{n}\, G^{-1}(p_\beta).$$

It is easily seen that (n,c) with the minimal n satisfying (B.1)-(B.2) is given by the intersection of the mappings $n \mapsto (\Phi^{-1}(1-\alpha) - \sqrt{n}\, G^{-1}(p_\alpha))$ and $n \mapsto (\Phi^{-1}(1-\beta) - \sqrt{n}\, G^{-1}(p_\beta))$, i.e., characterized by

$$\text{(B.3)}\quad \Phi^{-1}(1-\alpha) - \sqrt{n}\, G^{-1}(p_\alpha) \;=\; \Phi^{-1}(1-\beta) - \sqrt{n}\, G^{-1}(p_\beta).$$

This equation has a solution n since $G^{-1}(p_\alpha)$ is strictly smaller than $G^{-1}(p_\beta)$ (recall that G is strictly increasing). Plugging this n into (B.3) we obtain the corresponding c. Hence the optimal sampling plan (n, c) is indeed given by (2.8)-(2.9).

The derivation of the sampling plan (3.6)-(3.7) from (3.5) can be done along the same lines. Actually, this requires that the right-hand side of (3.5) is strictly decreasing (i.e., that the map $p \mapsto G_m(p)$ is strictly increasing) and that $G_m^{-1}(p_\alpha) < G_m^{-1}(p_\beta)$. Since the map $p \mapsto G_m(p)$ is a step function, the first requirement is violated and the second requirement may be violated. In particular, the equivalence of (2.5)-(2.6) to (B.1)-(B.2) (with G replaced by G_m) may be not anymore true, and the denominator of the right-hand side of (3.6) may equal zero. On the other hand, G_m uniformly converges almost surely to G (a Glivenko-Cantelli argument applies), and $G_m^{-1}(p)$ converges almost surely to $G^{-1}(p)$ for every p at which G is continuous and strictly increasing (cf., for instance, [17, Section 21]). Therefore the sampling plan (3.6)-(3.7) is at least asymptotically optimal and well-defined.

APPENDIX C: THE FUNCTIONAL DELTA METHOD

Here we briefly recall the essence of the functional delta method. For a more comprehensive exposition see, e.g., [11, 17]. Various applications of the delta methods can be found in [6, 13, 14] and in references cited therein. Let $(\mathbb{D}, \|.\|_{\mathbb{D}})$ and $(\mathbb{E}, \|.\|_{\mathbb{E}})$ be normed linear spaces, and $\mathbb{D}_f, \mathbb{D}_0 \subset \mathbb{D}$ and $\theta \in \mathbb{D}_f$. Suppose (T_n) is a sequence of $\mathbb{D}_f$-valued random elements and T is a $\mathbb{D}_0$-valued random element. Assume we know $\sqrt{n}(T_n - \theta) \xrightarrow{d} T$, but we are actually interested in the limit in distribution of $\sqrt{n}(f(T_n) - f(\theta))$ for some function $f : \mathbb{D}_f \to \mathbb{E}$. If $\mathbb{D} = \mathbb{E} = \mathbb{R}$ and f is differentiable with (Fréchet) derivative f', then one can use the Taylor expansion of order one to obtain

$$\begin{aligned}\sqrt{n}(f(T_n) - f(\theta)) &= \sqrt{n}\,(f'(\theta)(T_n - \theta) + o(T_n - \theta)) \\ &\approx f'(\theta)\sqrt{n}(T_n - \theta),\end{aligned}$$

and one would guess that this expression converges in distribution to $f'(\theta)T$. The delta method ([17, Theorem 3.1]) shows that this is in fact true. This method can be made rigorous also in the setting of general normed linear spaces $\mathbb{D}$ and $\mathbb{E}$. Then, of course, $f'(\theta)(.)$ has to be replaced suitably. It turns out that the Hadamard derivative $D^{Had}_{\theta;\mathbb{D}_0} f\ (.)$ of f at θ tangentially to $\mathbb{D}_0$ is the right analogue of $f'(\theta)(.)$ (note that in finite-dimensional spaces the Hadamard and Fréchet derivative coincide). In fact, provided the Hadamard derivative exists, the functional delta method ([17, Theorem 20.8]) implies

$$\sqrt{n}(f(T_n) - f(\theta)) \xrightarrow{d} D^{Had}_{\theta;\mathbb{D}_0} f\ (T).$$

For the notion of Hadamard differentiability see, for instance, [17, Section 20.2] or [6].

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