On Monitoring a Sequential Linear LS Residual Process for Integrated Errors

Ansgar Steland

Abstract: Motivated by applications in (econometric) time series analysis, we introduce and study a sequential residual process under the assumption that the error terms form a random walk. Based on the asymptotic distribution theory for that process we propose a monitoring procedure (control chart) which aims at detecting a change of the error terms from the random walk null hypothesis to the stationary case. The procedure is related to the well known Dickey-Fuller statistic. We provide new functional central limit theorems for the processes of interest under mild assumptions whcih cover dependent time series data. The results also yield the asymptotic distribution of the proposed monitoring procedure. The results are complemented by a numerical study investigating the performance properties of the method.

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1 Introduction

An interesting problem arising in time series analysis, particularly when dealing with econometric data, is to decide whether a trend-like behaviour can be explained by a deterministic trend component which is disturbed by stationary noise, or whether the random noise forms a random walk component (unit root case). In this paper we study a monitoring procedure based on a weighted version process version of the well-known Dickey-Fuller (DF) test, which is perhaps the most common unit root test. We provide functional central limit theorems for a least-squares residual process and the DF process based on that residual process yielding the asymptotic distribution of the proposed control chart.

Answering the unit root question has substantial implications for further analyses and interpretations. Indeed, even elementary statistics as sample means have different convergence rates and asymptotic laws if the process has a unit root instead of being stationary. The simplest model catching this idea is the AR(1) model $Z_t = \rho Z_{t-1} + u_t$, where $\rho \in (-1, 1]$ and $\{u_t\}$ are i.i.d. with $E(u_t) = 0$ and $E(u_t^2) \in (0, \infty)$. Then $\rho = 1$ corresponds to the random walk case, whereas for $\rho \in (-1, 1)$ the AR(1) equation has a strictly stationary solution. In this paper we will study a considerably more general nonparametric framework.

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Usually, so called unit-root tests are applied to test either the null hypothesis of a unit root against the alternative of stationarity, or vice versa. However, having in mind the importance of a correct answer, the application of sequential monitoring procedures is of great interest. These methods aim at quick detection of departures or changes from the so-called in-control model corresponding to the null hypothesis to a out-of-control model which is a member of the alternative hypothesis.

In this article we study a monitoring procedure (control chart) which aims to detect sequentially that the in-control model of a random walk with linear trend is not or no longer valid. As a class of out-of-control (alternative) models we assume that the time series can be described by a linear trend disturbed by stationary noise. We consider a truncated control chart where monitoring stops latest at the Tth observation. Indeed, in many real-world applications there exists a time horizon T, where a decision has to be made in any case or cost- and time-expensive analyses are conducted instead of cheap and fast monitoring schemes, which are applied between such periodical analyses.

Recall that a control chart is given by a stopping time, often of the form $S_T = \inf\{k \le t \le T : D_T(t) \in A\}$ for $\sigma(Y_s : s \le t)$-measurable control statistics $D_T(t)$, and a rejection region A. We use the convention $\inf \emptyset = \infty$. The pair (D_T, A) is chosen to ensure that the average run length (ARL), $E(S_T) = \int S_T dP$, satisfies $E_0 S_T \ge \xi$ for some prespecified value ξ, if $P = P_0$ where P_0 denotes the probability under the in-control model (null hypothesis), whereas $E_1 S_T = \int S_T dP_1$ should be small for any P_1 corresponding to an out-of-control model (alternative) of interest. Alternatively, one may design the procedure to control the type I error, i.e., $P_0(S_T \le T) \le \alpha$ for some given significance level $\alpha \in (0,1)$, and aim at high power, $P_1(S_T \le T)$, under alternatives P_1.

Let us assume that we observe sequentially a sequence $\{Y_t : t \ge 1\}$ of observations satisfying the model equation

$$Y_t = \beta_0 + \beta_1 t + \epsilon_t, \qquad t = 1, 2, \dots \tag{1}$$

Here β_0 and β_1 are fixed but unknown parameters of the linear trend component, and $\{\epsilon_t\}$ is a mean-zero error process with $\epsilon_0 = 0$. Note that $\beta_0 = E(Y_0)$. We use the following nonparametric definitions. A time series $\{\epsilon_t\}$ with $E(\epsilon_t) = 0$ is called integrated of order 0, denoted by $\{\epsilon_t\} \sim I(0)$, if the partial sum process $T^{-1/2} \sum_{i=1}^{\lfloor Ts \rfloor} \epsilon_i$ converges weakly to $\eta B(s)$, as $T \to \infty$, where $\eta \in (0,\infty)$ is a constant and $\{B(s) : s \in [0,1]\}$ denotes a Brownian motion (Wiener process). Here we regard the processes as elements of the Skorokhod space $D[0,1]$ equipped with the Skorokhod metric d. Weak convergence is denoted by $\Rightarrow$. Further, a time series $\{\epsilon_t\}$ with $E(\epsilon_t) = 0$ is called integrated of order 1, denoted by $\{\epsilon_t\} \sim I(1)$, if the first order differences $\Delta\epsilon_t = \epsilon_t - \epsilon_{t-1}$ are $I(0)$, and $T^{-1/2}\epsilon_{\lfloor Ts \rfloor} \Rightarrow \eta B(s)$, as $T \to \infty$, for some $\eta \in (0,\infty)$. This means, if $\epsilon_t \sim I(1)$ we have a random walk representation $\epsilon_t = \sum_{i \le t} \eta_i$ for random variables $\{\eta_t\}$ satisfying a functional central limit theorem, $T^{-1/2} \sum_{i=1}^{\lfloor Ts \rfloor} \eta_i \Rightarrow \eta' B(s)$, as $T \to \infty$, for some $\eta' \in (0,\infty)$. Note that these definitions cover dependent time series without posing conditions on the degree of the dependence. Conditions for the validity of a functional central limit theorem to (scaled) Brownian motion in terms of moment and α-mixing conditions can be found to Herrndorf (1985).

Using these definitions we can formulate the sequential decision problem as follows. $\{Y_t\}$ is in control (satisfies the null hypothesis), if $H_0 : \{\epsilon_t\} \sim I(1)$. The process $\{Y_t\}$ gets out-of-control if for some change-point q the series $\{Y_t : t < q\}$ is $I(1)$, whereas after the change it is $I(0)$, i.e., $\{Y_t : t \geq q\} \sim I(0)$. Usually it is assumed that the change-point is given by

$$q = \lfloor T\vartheta \rfloor,$$

for some $\vartheta \in (0, \infty]$, where $\vartheta = \infty$ corresponds to the null hypothesis.

The basic idea of the approach studied here is to eliminate the linear time trend by calculating sequential least squares residuals. To these residuals we apply an appropriate control chart which is able to detect stationarity. We will use a (weighted) Dickey-Fuller control chart, cf. [11].

Let us briefly discuss some related literature. Nonparametric detection of changes in the mean of a random walk using kernel smoothers has been studied in [10], and the related problem for stationary weakly dependent error processes is treated in [8]. The Dickey-Fuller unit root test was proposed by [3], and studied for various (parametric) time series. Nonparametric procedures to detect unit roots and stationarity when there is no deterministic structure have been studied extensively by [12] and [13]. The related problem to detect a change in a linear regression was recently studied by [5]. The sequential process of ARMA(p, q) residuals has been studied by [1] and applied to detect changes in the innovations.

2 Sequential Linear LS Residual Process

In this section we consider sequential least squares residuals. In contrast to the classic case of recursive residuals, where only the current residual is calculated using an update formula and analysed by, e.g., CUSUM methods, we study the case where at each time point *all* residuals are calculated using the available data. To our knowledge that residual process has not been studied for the $I(1)$ case in the literature. If we denote by t the current time point, we have the observations $Y_1, \ldots, Y_t$ available to calculated estimates of the error terms $\epsilon_1, \ldots, \epsilon_t$. It is natural to estimate the parameters (β_0, β_1) by least squares. Define

$$\begin{aligned} \widehat{\beta}_{t1} &= \sum_{i=1}^{t} (Y_i - \overline{Y}_t)(i - \bar{i}_t) / \sum_{i=1}^{t} (i - \bar{i}_t)^2 \\ \widehat{\beta}_{t0} &= \overline{Y}_t - \widehat{\beta}_{t0} \bar{i}_t, \end{aligned}$$

where $\overline{Y}_t = t^{-1} \sum_{i=1}^{t} Y_i$ and $\bar{i}_t = t^{-1} \sum_{j=1}^{t} j$. Using these estimates we may calculate the residuals

$$\widehat{\epsilon}_s(t) = Y_s - \widehat{\beta}_{t0} - \widehat{\beta}_{t1} s, \qquad 2 \leq s \leq t.$$

For $T \geq t$ denoting the time horizon we introduce the sequential linear LS residual process

$$E_T(r,s) = T^{-1/2}\widehat{\epsilon}_{\lfloor Tr \rfloor}(\lfloor Ts \rfloor), \qquad 0 \leq r \leq s \leq 1.$$

Note that by defining $E_T(r,s) = E_T(s,s)$ for $s \leq r \leq 1$ we may regard E as a random element of the Skorokhod space $D([0,1]^2)$. Indeed, the process we will be interested in the next section depends only on $\{E_T(r,s) : 0 \leq r \leq s \leq 1\}$. We equip $D([0,1]^2)$ with the Skorokhod metric d. The following theorem provides an explicit and simple representation of the distributional limit of E in terms of Brownian motion.

Theorem 2.1. *Assume model (1). If* $\{\epsilon_t\} \sim I(1)$*, then*

$$E_T \Rightarrow \mathcal{E}, \qquad T \to \infty,$$

where the a.s. continuous stochastic process $\mathcal{E}$ *is given by*

$$\mathcal{E}(r,s) = \eta \left\{ B(r) + \left(\frac{6r}{s^2} - \frac{4}{s} \right) \int_0^s B(u)\, du + \left(\frac{6}{s^2} - \frac{12r}{s^3} \right) \int_0^s uB(u)\, du \right\}, \quad (2)$$

for $0 < r \leq s \leq 1$*, and* $\mathcal{E}(r,0) = 0$ *for* $r \in [0,1]$*. Further,* $\mathcal{E} \in C([0,1]^2)$ *w.p. 1.*

Proof. A routine calculation shows that for $t \geq 2$ and $2 \leq i \leq t$ the residual $\widehat{\epsilon}_i(t)$ is given by

$$\epsilon_i - \frac{4t+2}{t(t-1)} \sum_{j=1}^t \epsilon_j + \frac{6}{t(t-1)} \sum_{j=1}^t j\epsilon_j + i \left(-\frac{12}{t(t^2-1)} \sum_{j=1}^t j\epsilon_j + \frac{6}{t(t-1)} \sum_{j=1}^t \epsilon_j \right). \quad (3)$$

Hence, with $Z_T(s) = T^{-1}\epsilon_{\lfloor Ts \rfloor}$, $s \in [0,1]$, we have for $0 < r \leq s \leq 1$

$$\begin{aligned} & T^{-1/2}\widehat{\epsilon}_{\lfloor Tr \rfloor}(\lfloor Ts \rfloor) = T^{-1/2}\epsilon_{\lfloor Tr \rfloor} \\ & - \frac{(4\lfloor Ts \rfloor + 2)T}{\lfloor Ts \rfloor(\lfloor Ts \rfloor - 1)} \int_0^s Z_T(u)\, du + \frac{6T^2}{\lfloor Ts \rfloor(\lfloor Ts \rfloor - 1)} \int_0^s \frac{\lfloor Tu \rfloor}{T} Z_T(u)\, du \\ & - \frac{12\lfloor Tr \rfloor T^2}{\lfloor Ts \rfloor(\lfloor Ts \rfloor^2 - 1)} \int_0^s \frac{\lfloor Tu \rfloor}{T} Z_T(u)\, du + \frac{6\lfloor Tr \rfloor}{\lfloor Ts \rfloor(\lfloor Ts \rfloor - 1)} \int_0^s Z_T(u)\, du. \end{aligned}$$

By the Dudley/Skorokhod/Wichura representation theorem (Skorack and Wellner (1986), Th. 4, p.47, and Remark 2, p.49) there exist equivalent processes, again denoted by Z_T and B, such that

$$\| Z_T - \eta B \|_\infty \to 0, \sup_{s \in [0,1]} \left| \int_0^s Z_T(u)\, du - \eta \int_0^s B(u)\, du \right| \to 0,$$

and

$$\sup_{s \in [0,1]} \left| \int_0^s \frac{\lfloor Tu \rfloor}{T} Z_T(u)\, du - \eta \int_0^s uB(u)\, du \right| \to 0$$

as $T \to \infty$, a.s. Noting that the above expression for $T^{-1/2}\widehat{\epsilon}_{\lfloor Tr \rfloor}(\lfloor Ts \rfloor)$ is a linear combination of Z_T, $\int_0^s Z_T(u)\,du$, and $\int_0^s (\lfloor Tu \rfloor / T) Z_T(u)\,du$, with coefficient functions from the class $C([0,1]^2)$, we may conclude weak convergence to the process

$$B(r) - \frac{4\eta}{s}\int_0^s B(u)\,du + \frac{6\eta}{s^2}\int_0^s uB(u)\,du - \frac{12r}{s^3}\int_0^s uB(u)\,du - \frac{6r\eta}{s^2}\int_0^s B(u)\,du,$$

which equals $\mathcal{E}$. Clearly, $\mathcal{E}$ is continuous w.p. 1. □

As a preparation for the next section we have to show that the first order differences of the residuals,

$$\widehat{u}_t(\lfloor Ts \rfloor) = \widehat{\epsilon}_t(\lfloor Ts \rfloor) - \widehat{\epsilon}_{t-1}(\lfloor Ts \rfloor), \quad t = 2, \dots, \lfloor Ts \rfloor,$$

yield consistent estimates for the first order differences $\Delta\epsilon_t = \epsilon_t - \epsilon_{t-1}$ of the error terms. Indeed, a stronger result providing the (uniform) convergence rate and the asymptotic distribution of the fluctuations of $\widehat{u}_t$ around $\Delta\epsilon_t$ can be shown.

Theorem 2.2. *Uniformly in each metric metrizing weak convergence in* $(D([0,1]^2), d)$ *we have*

$$T^{1/2}\{\widehat{u}_t(\lfloor Ts \rfloor) - \Delta\epsilon_t\} \Rightarrow \frac{6\eta}{s^2}\int_0^s B(u)\,du - \frac{12\eta^2}{s^2}\int_0^s uB(u)\,du,$$

as $T \to \infty$, *i.e., in particular*

$$|\widehat{u}_t(\lfloor Ts \rfloor) - \Delta\epsilon_t| = O_P(T^{-1/2}) = o_P(1).$$

Proof. First note that $\widehat{u}_t(\lfloor Ts \rfloor) - \Delta\epsilon_t$ does not depend on t. Hence, it suffices to show the assertion for a fixed t. Due to (3) we have

$$\widehat{u}_t(\lfloor Ts \rfloor) - \Delta\epsilon_i = \frac{6}{\lfloor Ts \rfloor(\lfloor Ts \rfloor - 1)}\sum_{j=1}^{\lfloor Ts \rfloor}\epsilon_j - \frac{12}{\lfloor Ts \rfloor(\lfloor Ts \rfloor^2 - 1)}\sum_{j=1}^{\lfloor Ts \rfloor} j\epsilon_j.$$

Again, since $T^{-5/2}\sum_{j=1}^{\lfloor Ts \rfloor} j\epsilon_j \to \eta\int_0^s uB(u)\,du$ and $T^{-3/2}\sum_{j=1}^{\lfloor Ts \rfloor}\epsilon_j \to \eta\int_0^s B(u)\,du$ in the sense of weak convergence, we may assume that the convergence also holds w.r.t. the supnorm. Therefore, we may conclude that $U_T(s) = T^{1/2}\{\widehat{u}_t(\lfloor Ts \rfloor) - \Delta\epsilon_t\}$ satisfies

$$\begin{aligned} U_T(s) &= \frac{6T^2}{\lfloor Ts(\lfloor Ts \rfloor - 1)} T^{-3/2}\sum_{j=1}^{\lfloor Ts \rfloor}\epsilon_j - \frac{12T^3}{\lfloor Ts(\lfloor Ts \rfloor^2 - 1)} T^{-5/2}\sum_{j=1}^{\lfloor Ts \rfloor} j\epsilon_j \\ &\Rightarrow \frac{6\eta}{s^2}\int_0^s B(u)\,du - \frac{12\eta^2}{s^2}\int_0^s uB(u)\,du, \end{aligned}$$

as $T \to \infty$, which verifies the assertion. □

3 A Control Chart

The basic idea of the monitoring procedure proposed here is to apply the Dickey-Fuller control chart to the sequential residuals defined and studied in the previous section. This means, at the current time t we have observed $Y_1, \dots, Y_t$ and calculate the residuals $\widehat{\epsilon}_1(t), \dots, \widehat{\epsilon}_t(t)$. To these residuals we apply a weighted version of the Dickey-Fuller statistic giving rise to the sequential weighted Dickey-Fuller (DF) residual process,

$$D_T(s) = \frac{\lfloor Ts \rfloor^{-1} \sum_{t=1}^{\lfloor Ts \rfloor} \widehat{\epsilon}_{t-1}(\lfloor Ts \rfloor) \Delta \widehat{\epsilon}_t(\lfloor Ts \rfloor) K(\{\lfloor Ts \rfloor - t\}/h)}{\lfloor Ts \rfloor^{-2} \sum_{t=1}^{\lfloor Ts \rfloor} \widehat{\epsilon}_{t-1}(\lfloor Ts \rfloor)^2}, \qquad s \in [0, 1].$$

Here K is a so-called kernel function satisfying

(K1) $K \geq 0$, $0 < \int K(z)\, dz < \infty$.

(K2) K is twice continuously differentiable and of bounded variation.

Common choices are the Gaussian and Epanechnikov kernel, which satisfy $K(z) \downarrow 0$ if $|z| \uparrow \infty$. $h = h_T$, $T \geq 1$, is a sequence of bandwidths satisfying

$$\zeta = \lim_{T \to \infty} T/h \in (1, \infty).$$

The associated control chart (stopping time) is given by

$$S_T = \inf\{k \leq t \leq T : D_T(t/T) < c\}.$$

for some control limit (critical value) c. Let us assume that the start of monitoring, k, is given by

$$k = \lfloor T\kappa \rfloor$$

for some $\kappa \in (0, 1)$. The following theorem can be used to choose the control limit c by relying on the asymptotic distribution of S_T/T, e.g. to ensure that $\lim_{T \to \infty} P_0(S_T \leq T) = \alpha$.

Theorem 3.1. *Assume model (1) and (K1), (K2). If*

(i) $\{\epsilon_t\} \sim I(1)$,

(ii) $E|\Delta\epsilon_t|^2 = \sigma^2 \in (0, \infty)$ *for all* t,

(iii) $\{|\epsilon_t| : t \geq 1\}$ *satisfies the weak law of large numbers, i.e.,* $\frac{1}{T} \sum_{t=1}^{T} \{|\epsilon_t| - E|\epsilon_t|\} \xrightarrow{P} 0$, *with* $\mu_\epsilon = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E|\epsilon_t| \in (0, \infty)$.

(iv) $\lim_{T \to \infty} T^{-1} \sum_{i,j=1}^{T} |Cov((\Delta\epsilon_i)^2, (\Delta\epsilon_j)^2)| < \infty$,

then the following assertions hold.

(i) *The sequential weighted DF residual process converges weakly,*

$$D_T \Rightarrow \mathcal{D}(s), \qquad T \to \infty,$$

where the process $\mathcal{D}(s)$ *is given by*

$$\frac{\frac{s}{2}\{K(0)\mathcal{E}(s,s)^2 + \zeta \int_0^s \mathcal{E}(r,s)^2 K'(\zeta(s-r))\,dr - \sigma^2 \int_0^s K(\zeta(s-r))\,dr\}}{\int_0^s \mathcal{E}(r,s)^2\,dr}$$

for $s \in (0,1]$, *and* $\mathcal{E}$ *is defined in (2)*

(ii) *The normed DF stopping rule after detrending,* S_T/T, *converges in distribution,*

$$S_T/T \xrightarrow{d} \inf\{s \in [\kappa, 1) : \mathcal{D}(s) < c\},$$

as $T \to \infty$.

Proof. We give a sketch of the proof using the techniques developed and given in detail in [13]. Note that by definition of $\widehat{u}_i(t)$ we have $\widehat{\epsilon}_i(t) = \widehat{\epsilon}_{i-1}(t) + \widehat{u}_i(t)$ and

$$\widehat{\epsilon}_{i-1}(t)\widehat{u}_i(t) = (1/2)\{\widehat{\epsilon}_i(t)^2 - \widehat{\epsilon}_{i-1}(t)^2 - \widehat{u}_i(t)^2\}.$$

The last equation implies the following decomposition of the process D_T:

$$\widehat{D}_T(s) = (V_T(s) - R_T(s))/W_T(s),$$

where

$$\begin{aligned}
V_T(s) &= \frac{1}{2\lfloor Ts \rfloor} \sum_{t=1}^{\lfloor Ts \rfloor} \left(\widehat{\epsilon}_t(\lfloor Ts \rfloor)^2 - \widehat{\epsilon}_{t-1}(\lfloor Ts \rfloor)^2\right) K(\{\lfloor Ts \rfloor - t\}/h) \\
R_T(s) &= \frac{1}{2\lfloor Ts \rfloor} \sum_{t=1}^{\lfloor Ts \rfloor} \widehat{u}_t(\lfloor Ts \rfloor)^2 K(\{\lfloor Ts \rfloor - t\}/h) \\
W_T(s) &= \frac{1}{\lfloor Ts \rfloor^2} \sum_{t=1}^{\lfloor Ts \rfloor} \widehat{\epsilon}_{t-1}(\lfloor Ts \rfloor)^2
\end{aligned}$$

Consider $R_T(s)$. Note that Theorem 3.1 and the inequality

$$|\widehat{u}_t(\lfloor Ts \rfloor)^2 - (\Delta\epsilon_t)^2| \le |\widehat{u}_t(\lfloor Ts \rfloor) - \Delta\epsilon_t|(|\widehat{u}_t(\lfloor Ts \rfloor) - \Delta\epsilon_t| + 2|\Delta\epsilon_t|)$$

imply that

$$T^{-1} \sum_{t=1}^{T} |\widehat{u}_t(\lfloor Ts \rfloor)^2 - (\Delta\epsilon_t)^2| = O_P(T^{-1/2})\{T^{-1} \sum_{t=1}^{T} 2|\Delta\epsilon_t| + O_P(T^{-1/2})|,$$

which is $o_p(1)$, since $P(T^{-1}\sum_{t=1}^{T}|\Delta\epsilon_t|>\epsilon)\leq P(|\frac{2}{T+1}\sum_{t=0}^{T}\{|\epsilon_t|-E|\epsilon_t|\}|<\frac{\epsilon}{2}-\frac{2}{T}\sum_{t=0}^{T}E|\epsilon_t|)\to 0$, as $T\to\infty$, for sufficiently large ϵ, by (iii).

$$\left|R_T(s)-\lfloor Ts\rfloor^{-1}\sum_{i=1}^{\lfloor Ts\rfloor}(\Delta\epsilon_t(\lfloor Ts\rfloor))^2K(\{\lfloor Ts\rfloor-t\}/h)\right|=o_P(1).$$

Hence, as shown in [13], condition (ii) and (iv) imply that

$$\sup_{s\in[\kappa,1]}|R_T(s)-\mu(s)|\stackrel{L_2}{\to}0,$$

with $\mu(s)=\sigma^2/(2s)\int_0^s K(\zeta(s-r))\,dr$ and $\sigma^2=E(\Delta\epsilon_t)^2$. We shall now show that for any $(\lambda_1,\lambda_2)\in\mathbb{R}^2-\{(0,0)\}$

$$\lambda_1V_T(s)+\lambda W_T(s)=\phi(E_T)+o_P(1) \tag{4}$$

for some continuous functional $\phi:D([0,1]^2)\to D[0,1]$. Using integration by parts one may show as in [13] that uniformly in $s\in(0,1]$

$$\begin{aligned}V_T(s) &= \frac{K(0)\mathcal{E}^2(s,s)}{2s}+\frac{\zeta}{2s}\int_0^s E_T^2(r,s)K'(\zeta(s-r))\,dr+o_P(1)\\ &= \tau_1(\mathcal{E})(s)+o_P(1),\end{aligned}$$

where the functional $\tau_1:D([0,1]^2)\to D[0,1]$ is given by

$$\tau_1(z)(s)=\frac{1}{2s}\left\{K(0)z(s,s)^2-K(\zeta s)z(0,s)^2+\zeta\int_0^s z(r,s)^2K'(\zeta(s,r))\,dr\right\}$$

Further, by Theorem 2.1,

$$\frac{1}{\lfloor Ts\rfloor^2}\sum_{t=1}^{\lfloor Ts\rfloor}\widehat{\epsilon}_{t-1}(\lfloor Ts\rfloor)^2=\tau_2(E_T)(s)+o_P(1),$$

if $\tau_2(z)=s^{-2}\int_0^s z^2(r,s)\,dr$, $s\in(0,1]$, and $\tau_2(0)=0$, for any $z\in D([0,1]^2)$. Hence, (4) holds with $\phi=\lambda_1\tau_1+\lambda_2\tau_2$. Now the result follows by an application of the continuous mapping theorem. □

Remark 3.2. A sufficient condition for (iii) is that $\gamma(k)\downarrow 0$ if $|k|\uparrow\infty$, where $\gamma(k)=Cov(|\epsilon_1|,|\epsilon_{1+k}|)$, $k\in\mathbb{Z}$, see Brockwell and Davis (1991, Th. 7.1.1).

Remark 3.3. Note that $\mathcal{E}$ depends on η. Hence, the process $\mathcal{D}$ depends on the parameter $\vartheta=\eta/\sigma$. If $\{\Delta\epsilon_t\}$ are uncorrelated, we have $\vartheta=1$ and the limit process is distribution-free.

4 Numerical Analysis

We conducted a simulation study to investigate the statistical properties of the proposed monitoring procedure in terms of statistical power and conditional ARL given that the procedure gives a signal. Time series of length $T = 250$ were simulated using the model

$$Y_t = 0.5 + 0.25t + \epsilon_t, \qquad 0 \le t \le T$$

with $\epsilon_t = \sum_{i=1}^t u_i$, if $t < q$, and $\epsilon_t = \epsilon_{q-1} + \rho\epsilon_{t-1} + u_t$, if $t \ge q$, where $\{u_t\}$ are i.i.d. random variables with distribution $N(0,1)$. Note that for observations before the change-point q the error terms ϵ_t form a random walk, whereas after the change the error process is AR(1) with coefficient ρ. We studied the cases $\rho = 0.7$ and $\rho = 0$. To these time series we applied the control chart S_T with bandwidth $h = 50$. We considered the Gaussian kernel (unbounded support) and the Epanechnikov kernel (bounded support). To investigate power and conditional ARL we used simulated control limits for a nominal significance level of $\alpha = 0.05$.

Kernel	Change-point q	Rejection Rate		Conditional ARL	
		$\rho = 0$	$\rho = 0.7$	$\rho = 0$	$\rho = 0.7$
Gauss	25	0.947	0.556	69.6	101.2
	50	0.718	0.239	103.5	113.7
	75	0.490	0.114	133.4	114.7
	100	0.290	0.068	153.4	89.8
	∞	0.048	0.046	59.9	60.7
Epanechnikov	25	0.919	0.436	68.8	91.8
	50	0.684	0.170	102.2	100.8
	75	0.428	0.081	131.8	88.8
	100	0.240	0.062	142.2	71.7
	∞	0.051	0.055	54.8	53.9

The conditional ARL values for $q < \infty$ show that often the correct decision that there is a change can be made very early. As expected, a change to i.i.d. is detected extremely well, whereas detection of a change to a AR(1) series with coefficient 0.7 is harder, but still satisfactory.

5 Concluding Remarks

We studied the problem to detect that the error terms in a time series with deterministic trend are $I(0)$ instead of $I(1)$ for the special case of a linear time trend. Here explicit and simple representations of the limit process can be derived. Overall, the results of the Monte Carlo are promising w.r.t. the applicability of the procedure and, overall, confirm the theory developed in this paper.

References

[1] J. Bai. Weak convergence of the sequential empirical process of residuals in ARMA models. *Annals of Statistics*, 22:2051-2061.

[2] P.J. Brockwell and R.A. Davis (1991). *Time Series: Theory and Methods.* Springer, New York.

[3] D.A. Fuller and W.A. Fuller. Distribution of the estimate for autoregressive time series with a unit root. *Journal of the American Statistical Association*, 74:427-431, 1979.

[4] N. Herrndorf. A functional central limit theorem for strongly mixing sequences of random variables. *Probability Theory and Related Fields*, 69:541-550, 1985.

[5] L. Horváth, M. Hušková, P. Kokoszka, and J. Steinebach. Monitoring changes in linear models. *Journal of Statistical Planning and Inference*, 126:225–251, 2004.

[6] W. Krämer and W. Ploberger. The CUSUM test with OLS resdiduals. *Econometrica*, 60:271-285, 2004.

[7] G.R. Shorack and J.A. Wellner (1986). *Empirical Processes with Applications to Statistics.* Wiley, New York.

[8] A. Steland. Sequential control of time series by functionals of kernel-weighted empirical processes under local alternatives. *Metrika*, 60:229-249, 2004.

[9] A. Steland. Optimal sequential kernel smoothers under local nonparametric alternatives for dependent processes. *Journal of Statistical Planning and Inference*, 132:131-147, 2005a.

[10] A. Steland. Random walks with drift - A sequential view. *Journal of Time Series Analysis*, 26:917-942, 2005b.

[11] A. Steland. A bootstrap view on Dickey-Fuller control charts for AR(1) series. *Austrian Journal of Statistics*, in press, 2006a.

[12] A. Steland. Monitoring procedures to detect unit roots and stationarity. *Econometric Theory*, in Revision, 2006b.

[13] A. Steland. Detecting unit roots in time series using weighted Dickey-Fuller processes. *Submitted*, 2006c.

Ansgar Steland: RWTH Aachen University, Institute of Statistics, Wüllnerstr. 3, Aachen, 52056, Germany, steland@stochastik.rwth-aachen.de